ABSTRACT

WHY THALES KNEW THE PYTHAGOREAN THEOREM

Once we accept that Thales introduced geometry into Greece, having traveled to Egypt, as Proclus reports on the authority of Eudemus, who also credits Thales with a number of theorems, we understand that Thales was making geometrical diagrams. From where did he see such diagrams? Egypt is one place, having measured the height of a pyramid there. Diagrams that reflect measurements when the shadow was equal to its height, and un-equal but proportional – the doxography credits him with both techniques -- suggest that Thales understood similar triangles – ratios, proportions, and similarity.

We begin with the diagrams associated with the theorems, and place them next to the ones that reflect the measurements of the pyramid and distance of a ship at sea, Then, we introduce, on the authority of Aristotle, that Thales posited an *archê*, a principle, from which all things come, and back into which all things return upon dissolution – there is no change, only alteration – a big picture begins to form. Suppose, then, Thales investigated geometry, whether or not they started as practical exercises, as a way to solve the metaphysical problem of explaining HOW this one underlying unity could appear so divergently, modified but not changing? Geometry offered a way to find the basic figure into which all other figures resolve, that re-packed and re-combined, was the building block of all other appearances. We might see a lost narrative of the relation between philosophy and geometry. That narrative is preserved later by Plato at *Timaeus* 53C and following: the construction of the cosmos out of right triangles.

There are two proofs of the Pythagorean theorem, not one, preserved by Euclid. The one we learned in school, if we learned it at all, was I.47 that Proclus reports was Euclid's own invention. But, the *other* one, in book VI.31, by similar figures, by ratios and proportions, plausibly points back to Thales himself, perhaps taken up and perfected in proof by Pythagoras and the Pythagoreans. That proof shows that the right triangle is the fundamental geometrical figure, that expands or contracts in a pattern that came to be called continuous proportions. The argument that Thales knew the hypotenuse theorem is that, surprisingly, this was what he was looking for to explain HOW a single unity could appear so divergently, altering without changing. The plausibility rests on following the diagrams as evidence.

WHY THALES KNEW THE PYTHAGOREAN THEOREM[1]

A
Introduction and Overview

On the one hand, Aristotle identifies Thales as the earliest philosopher of whom he knows, for Thales posited an ἀρχή, a principle, an underlying unity from which all things come forth and back into which they return upon dissolution – this is *metaphysics*. Thales, and the other early philosophers, according to Aristotle, posited some underlying cosmic unity that alters without changing; thus all appearances are only modifications of this underlying unity. On the other hand, from Proclus, on the authority of Eudemus, Aristotle's student, we learn that Thales went to Egypt, introduced geometry into Greece, and is connected with specific theorems that he proved, noticed, observed, or stated.[2] What has been missing in our understanding of Thales – what I shall refer to as the "lost narrative" -- is ***the connection between philosophy and geometry, the metaphysical context in which geometry unfolded***. Because the reception of Thales' activities of measuring heights and distances relegated them to the sphere of practical problem-solving, the underlying metaphysical context of geometry has escaped notice. But the *project* contained in the narrative of constructing the cosmos out of right triangles, preserved by Plato in the *Timaeus* 53Cff, plausibly traces back to the Pythagoras and the Pythagoreans, and before them to Thales himself.

Thales was thinking in terms of geometrical diagrams. Since Thales stands as the first Greek geometer of whom we know, from where could he have seen them, and so have been inspired to think in terms of them? From where could he have learned from them? To this broader context I turn first. He could have seen them had he visited Mesopotamia or had others in his retinue brought a knowledge of them to Miletus, and he could have seen them when he was in Egypt. He could also have seen them in informal sketches by the Greek architect building monumental temples in his very backyard.

B
The Problems Concerning Geometrical Diagrams

To investigate this missed connection between geometry and metaphysics, we shall now explore the background of geometrical diagram-making in the context of which Thales' innovations are illuminated. Because there are no surviving diagrams by Thales, only references to theorems, the argument for the plausibility that Thales made them is to show that we do have evidence of diagram-making in Mesopotamia whose influence may have been communicated to Ionia, from Egypt where Thales measured the height of pyramid, and also evidence in eastern Greece – Ionia -- where Thales lived. But let me try to be very clear on this point: it makes no sense to credit Thales with the isosceles triangle theorem, for example -- that if a triangle has two sides of equal lengths, then the angles opposite to those sides are equal – without crediting him with making diagrams! And there are no good reasons to deny Thales credit for such a theorem. Furthermore, I will argue that Thales plausibly knew an interpretation of the Pythagorean theorem, and this is because, together with the doxographical reports about Thales efforts in geometry, the knowledge of that theorem turned out to be fundamental to his metaphysics. An interpretation of it was certainly known at least a millennium before Thales' time in Mesopotamia, and it is possible that some interpretation of it was known in Egypt, but my argument is that the case for Thales' knowledge of it, and Pythagoras and the Pythagoreans in turn, emerged from a completely different context of inquiries.

Despite the fact that the origins of geometry in ancient Greece have been the subject of endless controversies, a new light into the early chapters can shine by taking a new approach to understanding the 'Pythagorean theorem'. Despite the fact that a consensus among scholars of ancient philosophy has consolidated around Burkert's claim that Pythagoras himself probably had no connection with the famous theorem,[3] the knowledge of the hypotenuse theorem belongs to an early insight in this unfolding story, customarily placed in the fifth century. I will argue that its discovery among the Greeks[4] is plausibly earlier, as early as the middle of the sixth century B.C.E., regardless of whose name we attach to it, though the case I argue for is that Thales plausibly knew an interpretation of it. If Thales knew it, then the members of his school knew it, and if they knew it, it makes no sense to deny Pythagoras knowledge of it, whether he learned it from Thales, Anaximander, or a member of their retinue. When we see why and how it was plausibly discovered in the Greek tradition at such an early date, we get a new insight into the origins of geometry there. To explore this matter, we first have to come to grasp what the famous theorem means, what it could have meant to the Greeks of the Archaic period, had they known it then, and recognize that these answers might be quite different. For from the standpoint of trained mathematicians, we might get some kinds of answers, and from the context of the Greeks of the 6th century when there were no geometry texts, and geometrical knowledge was for them in its infancy, we might get some other kinds of answers. We have got to figure out what we are looking for.

It is my contention, and the argument of my recent book,[5] that the theorem was discovered by the Greeks of the 6th century who were trying to resolve a *metaphysical* problem. It was not originally a theorem discovered in the context of a mathematics seminar. The whole picture of understanding the hypotenuse theorem, and hence the origins of Greek geometry, has been misconceived. Geometry was originally the handmaiden to metaphysics.

Recent studies by Netz, for example, have directed us to see the origins of geometry in the context of geometrical diagrams, and specifically lettered diagrams, and the prose writing that is connected inextricably with them.[6] At one point Netz argues that "Our earliest evidence for the lettered diagram comes from outside mathematics proper, namely from Aristotle."[7] But, the dating for the earliest geometrical diagrams Netz fixes by speculative inference just after the middle of the 5th century B.C.E. with the "rolls" of Anaxagoras, Hippocrates, and Oinopides; these are the earliest "publications" referred to by later writers where diagrams are mentioned to be contained in them, though none of them survive from the fifth century. When Netz tries to disentangle the testimonies of Simplicius (6th century C.E.), who refers to Eudemus' testimony (4th century B.C.E.) about Hippocrates' efforts in squaring the *lunulae* (5th century B.C.E.), he cautions and advises us about "layering" in these testimonies to sort out who said what: Simplicius had his own agenda in selecting the pieces in Eudemus' report about Hippocrates, as well as did Eudemus in presenting and discussing them.[8] Netz' general conclusion is that our view of even the fifth century is distorted as a consequence of disciplinary boundaries set by Aristotle and others in the fourth century upon whom the doxographical reports rely, who separate out the world into disciplinary divides of their own choosing and for their own purposes. Ironically, Netz has done much the same thing without grasping adequately that he has done so. His case about the origins and early development of geometry focuses on geometrical diagrams, and specifically lettered diagrams, to identify the earliest episodes in Greek mathematics; but his approach defines away evidence that changes the story itself. Netz focuses on diagrams in a mathematically "pure" sense, not applied. Netz' mathematician wants to know when the diagram has a reality all of its own, not as an architectural illustration of a building but is the building itself. Let me grant out of hand that regarding the diagram as an object onto itself – like grasping a Platonic form -- reflects a different, special kind of consciousness; it is the mathematical object *per se* within the disciplinary guidelines for the mathematician, and exists in a realm all its own. And Netz is certainly entitled to focus on this moment and tell a story spring boarding from it. But it just so happens that we have evidence of lettered diagrams *almost a century earlier* than Anaxagoras and Hippocrates, and two centuries earlier if we fix on Netz' claim that it is Aristotle who provides the earliest evidence; the evidence for a lettered diagram traces to the middle of the 6th century in Pythagoras' backyard itself – from the tunnel of Eupalinos of Samos[9] – and evidence from other building sites such as the nearby monumental temple of Artemis at Ephesus dating even earlier to the middle of the 7th century B.C.E.[10] that suggest that thinking and reflecting on geometrical diagrams was likely widely acknowledged and practiced much earlier when architects/engineers sought to solve their problems, guide the workers producing the architectural elements, or explain their solutions to others. Let us consider how this evidence changes the story of the origins of Greek geometry.

Netz' argument is that "…the mathematical diagram did not evolve as a modification of other practical diagrams, becoming more theoretical until finally the abstract geometrical diagram was drawn. Mathematical diagrams may well have been the first diagrams…not as a representation of something else; it is the thing itself."[11] Now it should be pointed out that Netz acknowledges that "…but at first, some contamination with craftsman-like, the 'banausic', must be hypothesized. I am not saying that the first Greek mathematicians were e.g. carpenters. I am quite certain that they were not. But they may have felt uneasily close to the banausic…"[12] Those fifth century "mathematicians" to whom Netz is referring might well have felt uneasily close to the banausic, but probably not as much as modern mathematicians do who distinguish sharply between pure and applied mathematics by their own modern disciplinary divides. And when we see that geometrical diagrams were explored already long-before the middle of the 6th century, it should be clearer yet that the distinction of "pure" vs "applied" is perfectly misleading to how these investigations started. The point is that the evidence we do have shows that early forays in Greek geometry in the Archaic period were connected to architectural and building techniques, engineering problems, and practical, technical activities such as measuring the height of a pyramid by its shadows, measuring the distance of a ship at sea. It is my argument that these practical applications of geometrical techniques led to and involved the making of geometrical diagrams and reflecting further on them, at least as early as the time of Thales; these practical applications urged and invited Greeks of the Archaic period to think about the relations among rectangles, squares, triangles, and circles. Netz contends that "What is made clear by [his] brief survey is that Greek geometry did not evolve as a reflection upon, say, Greek architecture."[13] At all events, it is my contention that the mathematical diagram did indeed evolve as modifications of other practical diagrams, especially from architecture and engineering projects, since these diagrams proliferated for more than a century before the time of Anaxagoras and Hippocrates, and more than two centuries before the time of Aristotle. This is how the evidence changes the story.

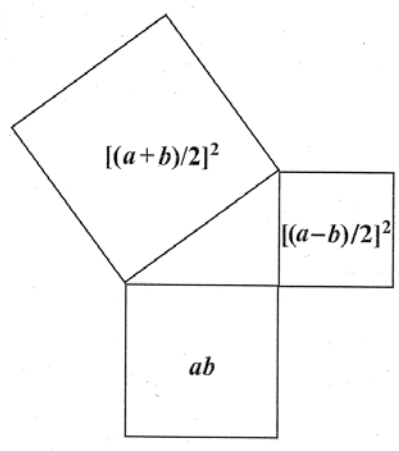
The main theme missing from the scholarly literature in the narrative I am going to unfold is that the connection that made all this possible is the principle that underlies the *practical diagram* – and I will argue that we can begin with Thales and his contemporaries. The practical diagram was an early form of "proof," a making visible of a set of connections that was being claimed. The diagram and its explanation may well fall short of the kinds of rigorous deductions we find a century or more later, but as von Fritz expressed it,[14] it was persuasive for an audience of the Greeks by the standards of that time. The revealing point of the practical diagram was that the cosmos could be imagined in such a way that an underlying, orderly structure – a non-changing frame of relations – was discoverable, and could illuminate our everyday experience that did not at first offer clues about it. The limestone hill through which Eupalinos' tunnel is cut has no perfectly straight lines, and yet by imagining and planning a geometry of straight lines, the digging of the tunnel from two sides was achieved successfully. As Kienast argued, Eupalinos had to have made not only a lettered diagram but also a scaled-measured diagram to achieve his result, and the evidence for it is still on the tunnel walls! It is this sort of evidence that undermines the kind of narrative Netz embraces so far as it reveals the origins of geometrical thinking for the Greeks; by focusing on the diagram as a thing-in-itself, what is missed is the fact that lettered diagrams did not start this way but rather began by presupposing that an intelligible, non-changing structure underlies and is relevant to the world of altering appearances that we experience, that the diagram contains an insight into the world of appearances. There are deep truths that lie behind the appearances, and geometry offered a way to reveal and express them. It is only because the architectural diagram – the practical diagram – "represents the building" that geometry came to be seen as offering a new and special window into the structure that underlies it. Thus, because the practical diagram pointed to a hidden underlying structure that nevertheless found application to our world of ever-changing experiences, it came to acquire a *metaphysical* relevance.

C
Diagrams and Geometric Algebra: Babylonian Mathematics[15]

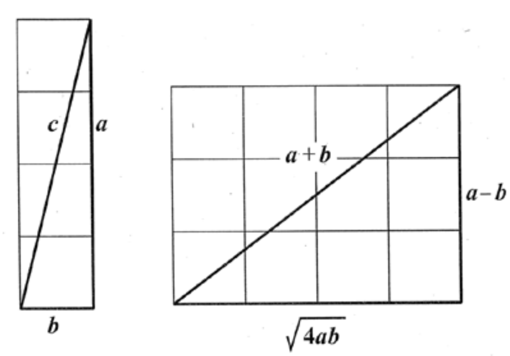
The thesis that Babylonian mathematics informed, stimulated, and influenced Greek geometry has had supporters and currency. One way to see this discussion is to consider the theme of "geometric algebra," that Mesopotamian mathematics from

Cuneiform tablets –from the early days of the Sumerians to the fall of Babylon in 539 B.C.E. -- supplied evidence for what the historian of mathematics Otto Neugebauer described as quadratic algebra.[16] And when scholars reviewed Euclid's geometry they came to the hypothesis that some of the *Elements* were geometric solutions to these very same problems in quadratic algebra. These geometrical solutions came to be termed "geometric algebra," and this was the basis for suggesting Mesopotamian influence and stimulation. While there is no evidence to discount possible exchange of mathematical knowledge and stimulus, there is also no direct evidence for it. The argument for substantial Babylonian influence on the character of Greek *geometry* seems to me to be tenuous; but positively and more importantly for our project, what the evidence does show is that had Thales and his 6th century compatriots been aware of what the Cuneiform tablets supply, there was robust evidence for thinking about spatial relations through diagrams.[17]

The main positive point I wish to emphasize here about Babylonian mathematics is that more than a thousand years before the time of Thales and Pythagoras, we have evidence for geometrical diagrams that show the visualization of geometrical problems. Recently, this point had been made emphatically: "that [an analysis of both Egyptian and Babylonian texts show] geometric visualization was the basis for essentially all of their mathematics" because visualization is simply a natural and intuitive way of doing mathematics.[18] Below we have visualization of right-angled triangles and the 'Pythagorean Theorem': below left, we have a visualization of Old Babylonian problem text YBC 6967; and below, right, a visualization of the "Pythagorean Theorem" in the Babylonian tradition, and finally, further below, the OB tablet YBC 7289. And note, these geometrical diagrams contains *numbers*; there are <u>no</u> numbers on any of Euclid's geometrical diagrams, and no discussion on numbers at all through the first six books of Euclid.

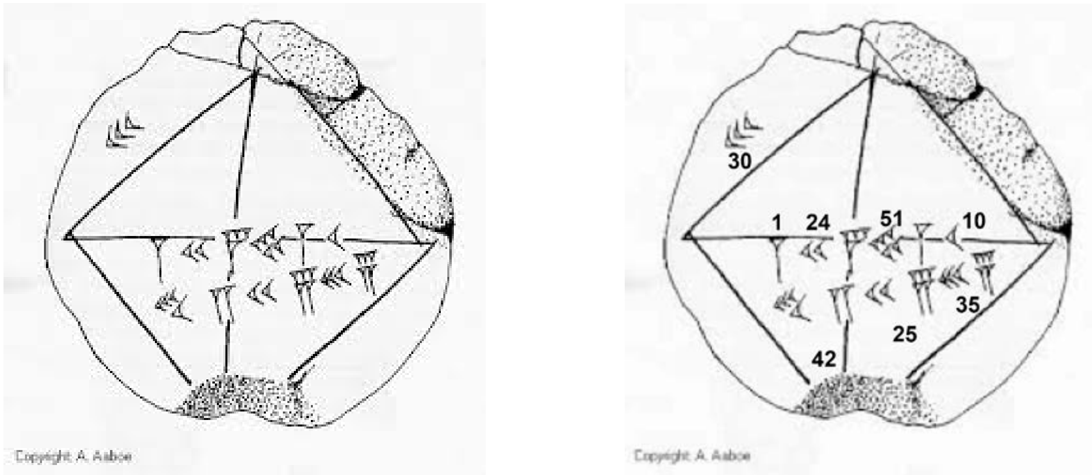


Right Triangle visualization of Old Babylonian problem text YBC 6967[19]



"The Babylonian Theorem"
For any Right triangle (a, b, c) it is possible to construct another right triangle with sides: √4ab, a-b, a + b[20]

Fig. 1



After YBC 7289
the ‘Pythagorean Theorem’[21]

The diagonal displays an approximation of the square root of 2 in four sexagesimal figures, 1 24 51 10, which is good to about six decimal digits. $1 + 24/60 + 51/60^2 + 10/60^3 = 1.41421296...$ The tablet also gives an example where one side of the square is 30, and the resulting diagonal is 42 25 35 or 42.4263888.

Fig. 2

Thus, we have a plentiful supply of evidence from Mesopotamia for geometrical diagrams long before the time of Thales, and even contemporaneous with him, and as we shall consider next, evidence for geometrical diagrams from Egypt, a location where we can certainly place him. The Babylonian evidence shows an awareness of the relationship between the side lengths of a right triangle. But it is my contention that the discovery of the hypotenuse theorem *in the Greek tradition* was a consequence of thinking through a *metaphysical* problem, of seeking to find the fundamental geometrical figure to which all rectilinear figures dissect – the right triangle – and we have no evidence that the right triangle played such a role in any metaphysical speculations from either Mesopotamia or Egypt.

D
Diagrams and Ancient Egyptian Mathematics: What Geometrical Knowledge Could Thales have Learned in Egypt?

Thales plausibly learned or confirmed at least three things about geometry from his Egyptian hosts and all of them involved diagrams: (1) formulas and recipes for calculating the area of rectangles and triangles, and volumes, and the height of a pyramid (i.e. triangulation); (2) from the land-surveyors he came to imagine space as flat, articulated by rectilinear figures all of which were reducible ultimately to triangles to determine their area; (3) watching the tomb painters and sculptors, geometrical similarity: the cosmos could be imagined -- flat surfaces and volumes -- articulated by squares, and each thing can be imagined scaled-up of a smaller version of it. The cosmos could be imagined – small world and big world – to share the same structure; the big world is a scaled-up version of the small world with which it is geometrically similar.

It is my purpose to set out a plausible case for what Thales could have learned by exploring some possibilities, but at all events I alert the reader to the speculative nature of my case. While the duration of time in which the surveyors worked, the architects built pyramids and temples, and the mathematics texts were produced spanned thousands of years, the surviving evidence from Egypt is exiguous; only a few papyri and rolls have so far been discovered. Clearly, there must have been so much more evidence, but in its absence I can only explore some possibilities. I regard that some of the evidence suggests that geometrical problems were to be solved graphically, that is visually. And for this study I try to answer one central question generated from RMP #51: "What is a Triangle's Rectangle?"

Eudemus claims that Thales traveled to Egypt and introduced geometry into Greece. He never says that Thales learned all the principles of it there, nor does he claim that Thales learned the deductive method of proof from Egyptian sources. But Eudemus claims that Thales discovered many things about geometry, and in turn, investigated these ideas both in more general or abstract ways, and also in empirical or practical ways:

> Θαλῆς δὲ πρῶτον εἰς Αἴγυπτον ἐλθὼν μετήγαγεν εἰς τὴν Ἑλλάδα τὴν θεωρίαν ταύτην καὶ πολλὰ μὲν αὐτὸς εὗρεν, πολλῶν δὲ τὰς ἀρχὰς τοῖς μετ᾽αὐτὸν ὑφηγήσατο, τοῖς μὲν καθολικώτερον ἐπιβάλλων, τοῖς δὲ αἰσθητικώτερον.
>
> Thales, who traveled to Egypt, was the first to introduce this science [i.e. geometry] into Greece. He discovered many things and taught the principles for many others to those who followed him, approaching some problems in a general way, and others more empirically.

First of all, Eudemus places together Thales travel to Egypt with his introduction of geometry into Greece. This suggests that there was some geometrical knowledge he learned there. The scholarly literature, even as recently as Zhmud's new study,[22] has downplayed such a connection. This is because Zhmud, and others, have come to regard "deductive proof" as central to Greek geometry, and to the role that Thales contributed to this direction; since there is no evidence that survives in the Egyptian mathematical papyri that shows deductive strategies, the contribution of Egypt to Greek geometry has been reduced in apparent importance. Zhmud's objections to Eudemus' claim that geometry was learned and imported by Thales from Egypt are twofold: (i) those like Thales had no way to communicate with the Egyptian priests (the Greeks never took an interest in learning foreign languages), and (ii) even if he could, there would be nothing for him to have learned about deductive reasoning since what we do have are only formulas and recipes for practical problem solving without the explicit statement of any general rules to follow.[23]

It seems to me that Herodotus had it right when he claimed that the Greeks learned geometry from the Egyptians through land-surveying, in addition to what they could have learned from priests and merchants about the contents of the mathematical papyri. Herodotus informs us that after the Milesians assisted Pharaoh Psammetichus [i.e. Psamtik, c. 664-610] to regain his kingship and so to once again re-unite upper and lower Egypt under a single Egyptian ruler, the grateful pharaoh allowed the Milesians to have a trading post in Naucratis, in the Nile delta; this location was very close to the pharaoh's capital in Sais.[24] Herodotus tells us that the pharaoh sent Egyptian children to live with the Greeks and so learn their language, and so facilitate communication as their translators and interpreters.[25] Later, when Amasis ascended to the throne (c. 570), Herodotus tell us that these Greeks were re-settled in the area of Memphis, that is south of Giza, and served as his own bodyguard; Amasis needed to be protected from his own people.[26] Since the picture we get of the Greeks in Egypt shows them to be embraced warmly and gratefully by the pharaohs, when Thales came to Egypt it makes sense that he traveled with the continuing assistance and support of the Egyptian authorities, conversing with the Egyptian surveyors, priests, and merchants through these translators and interpreters.

It also seems to me that what happened in the originating stages of geometry in Greece is close to the one offered by van der Waerden in *Science Awakening*. In his estimation, Thales returned from Egypt and elsewhere with practical formulas and recipes and sought to prove them; this is how Thales, in his estimation, stands at the beginnings of geometry in Greece. What I add to van der Waerden's narrative is this: Thales returned to Miletus and shared with his compatriots what he learned from the surveyors, priests, and the practical merchants who used the formulas and recipes contained in the Rhind Mathematical Papyrus [RMP] and Moscow Mathematical Papyrus [MMP] (both dating to the Middle Kingdom, c. 1850 B.C.E. that is, more than a thousand years *before* Thales came to Egypt. The RMP Problems 41 – 46 show how to find the volume of both cylindrical and rectangular based granaries; Problems 48–55 show how to compute an assortment of areas of land in the shapes of triangles and rectangles; Problems 56-60 concern finding the height or the *seked* (i.e. inclination of the face) of pyramids of a given square base. Thus the problems included formulas and recipes that showed how to divide 7 loaves of bread among 10 people [Problem 4], how to calculate the volume of a circular granary that has a diameter of 9 and a height of 10 [Problem 41], how every rectangle was connected inextricably to triangles that were its parts [Problems 51 and 52] – and also from the land-surveyors. His compatriots must surely have been as intrigued as they were skeptical. We can imagine members of his retinue asking: "Thales, how do you know that this formula is correct?" "How can you be sure?" And his replies were the first steps in the development of "proofs." I invite any teacher who has ever taught mathematics or logic to reflect on his/her own experiences; in the process of explaining countless times to many students how to figure out problems and exercises, the teacher comes to grasp ever more clearly the general principles and consequently deductive strategies – so did Thales. The question of whether and to what extent Thales could have produced a formal proof, and thus the endless wrangling over whether and to what degree Thales invented, developed, and employed the *deductive method*, has bogged down the literature, fascinating a question as it may be. The way to understand what happened in these earliest chapters is to ask, instead, what might have been found convincing by Greeks of the 6th century B.C.E., regardless of whether the lines of thought would meet the formal requirements that we later find in Euclid. From priests and practical people who divided food among the workmen and stored grain, Thales learned these formulas and recipes. Also in these mathematical papyri were formulas and recipes for determining the area of a triangle, a rectangle, calculating the height of a pyramid from the size of its base and inclination (*seked*) of its face. It seems so plausible that Thales' measurement of a pyramid height by means of shadows was a response to learning how the Egyptians calculated pyramid height by their own recipes, and the ingenuity he brought to explore afresh these long held formulas.

The second part of Thales' instruction into geometry in Egypt came from the land surveyors; it is just this kind of knowledge to which Herodotus alludes.[27] Every year, after the Nile inundation, the surveyors carrying the royal cubit cord supervised by a priest who wrote down and recorded the measurements, returned to each man who worked the land his allotted parcel; just this scene is captured in the tomb of Menna in the Valley of the Nobles (below). This means that each man had returned to him, by re-surveying, a parcel of land of the same *area*. Taxes were assessed – the quantity of crops to be paid as tribute -- in terms of the area of land allotted to each man.[28] It is my contention that, not infrequently, the land was so eviscerated by the surging, turgid flow of the Nile flood that parcels of land, customarily divided into rectangles (*arouras*), had to be returned in other shapes but with the same area. Through land surveying, the Egyptians came to understand that every rectilinear figure can be reduced to the sum of triangles. The evidence for this claim for the calculation of a triangular plot of land is delineated by the RMP Problem 51 (and almost the same calculation at MMP 4, 7, and 17) and Problem 52 for the calculation of a *truncated triangle of land*, a trapezium; here we have a window into just this reality of reckoning parcels of land that were not the usual rectangles, and the developing sense of areal equivalences between different shapes. It was here that Thales plausibly learned or confirmed equality of areas between geometrical figures of different shapes. When is a square equal in area to a rectangle? When is any polygon equal to a rectangle? The answer to all these questions of areal equivalence, is driven by practical needs and shows that techniques of "land-measurement" (*geō + metria*) really do find application to the land itself; Thales was in a position to have learned or confirmed from the Egyptians *polygonal triangulation* -- that every rectilinear figure can be divided into triangles, and by summing up the areas of the triangles into which every figure dissects, the area of every polygon can be reckoned. From the countless times the surveyors re-surveyed the land, they developed an intuition for areal equivalences between shapes. It is my argument that the reduction of all figures to triangles is the quintessential metaphysical point that Thales learned or confirmed from Egyptian geometry whether or not it was an explicit part of Egyptian teaching – the formulas relating triangles and rectangles show that it was implied.

The third thing that Thales plausibly learned or confirmed in Egypt is the idea of geometrical similarity. "Similarity" is the relation between two structures with the same shape but of different size; moreover, similarity was the principle that underlies and drives the microcosmic-macrocosmic argument, that *the little world and big world share the same structure*. Consequently, a vision burgeoned that the cosmos is a connected whole interpenetrated throughout by the *same* structure, and by focusing on the little world, we get deep and penetrating insights into the big world of the heavens and cosmos itself, otherwise inaccessible. Whether the Egyptian tomb painters and sculptors grasped this principle and/or could articulate it, we cannot say with confidence, but their work displays it. And Thales might well have taken a lesson in geometrical similarity by watching them work.

We have abundant evidence left by ancient Egyptian tomb painters for the use of square grids, made with a string dipped into red ink, pulled taut and snapped against the wall,[29] and some ruled against a straight-edge. The evidence is robust from as early as the 3rd dynasty through Ptolemaic times. Below, left, we have the red grid lines still visible from the New Kingdom tomb of Ramose in the Valley of the Nobles; below, right, we have a figure on its original grid dating to the Saite period, contemporaneous with time of Thales' visit to Egypt.[30]

Photo by the author

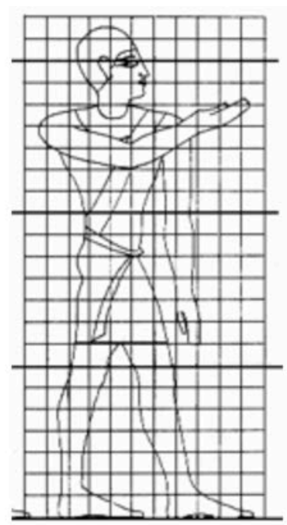
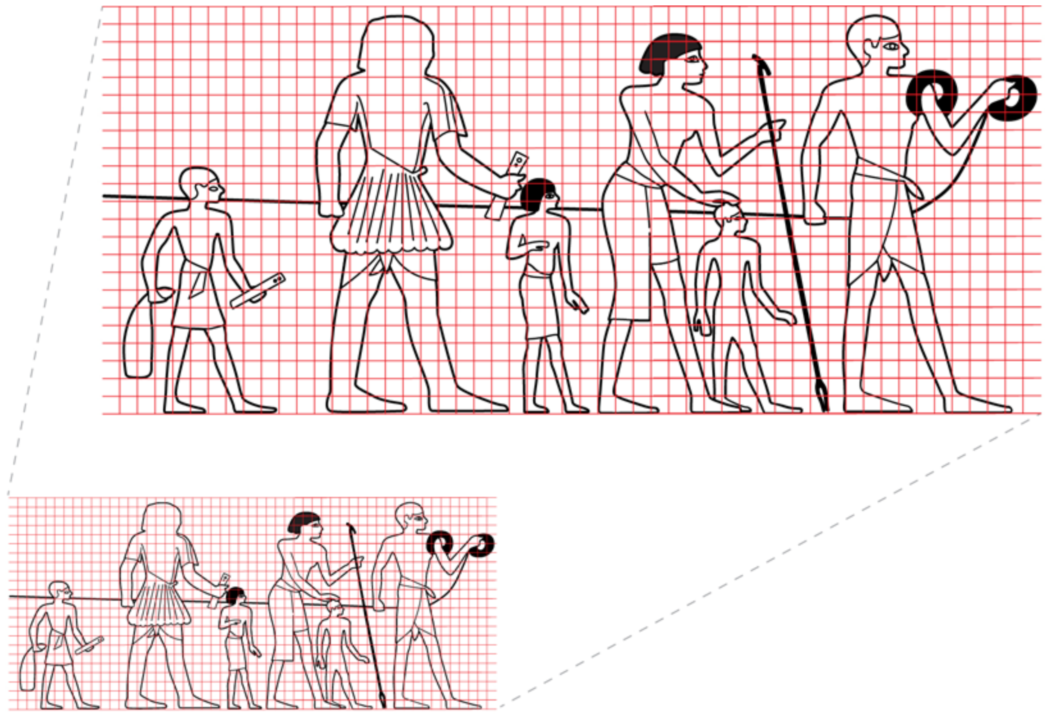


Fig. 3

Robins also includes an example of Pharaoh Apries (c.589-570 B.C.E.) standing between two deities with the original grid. Thus, within the frame of time that Thales visited Egypt, the grid system for painting was still in use.[31]

There has been debate about just how the grids were used; they may have been used to make a small sketch on papyrus or limestone, for example, and then, a larger square grid might have been prepared on the tomb wall and the painter would transfer the details in each little square to the larger squares, that is, the tomb painting would be a scaled-up version of the smaller sketch with which it was geometrically similar. "Both primary and secondary figures on grids may frequently have been the work of apprentices attempting to follow the proportion style of the masters; the grid was not only a method for design but also one for teaching the art of proportion by means of workshop production."[32] This idea is represented in the diagram directly below, after the tomb of Menna; here we have depicted the surveyors with the measured cord and the scribe (with ink block in hand) to record the results:



After the tomb of Menna in the Valley of the Nobles (Luxor, Egypt)

Fig. 4

We do have surviving examples of small sketches with red grids, such as an ostracon like that of New Kingdom Senenmut (below left),[33] or white board (below, right),[34] guiding the draftsman to produce human figures in appropriate proportions.

Owned by the British Museum

Fig. 5

Whether working on a small sketch or large tomb wall, both square grids would follow established rules of proportion, though the rules seem to have changed throughout the dynasties. Consider the illustration, below, where the rules established in the Older Canon are displayed: for standing figures 18 squares from base to the hairline, and proportional rules for seated figures.[35] In the New Kingdom, 19 grid squares to the neck and 22.5 to the top of the crown.[36]

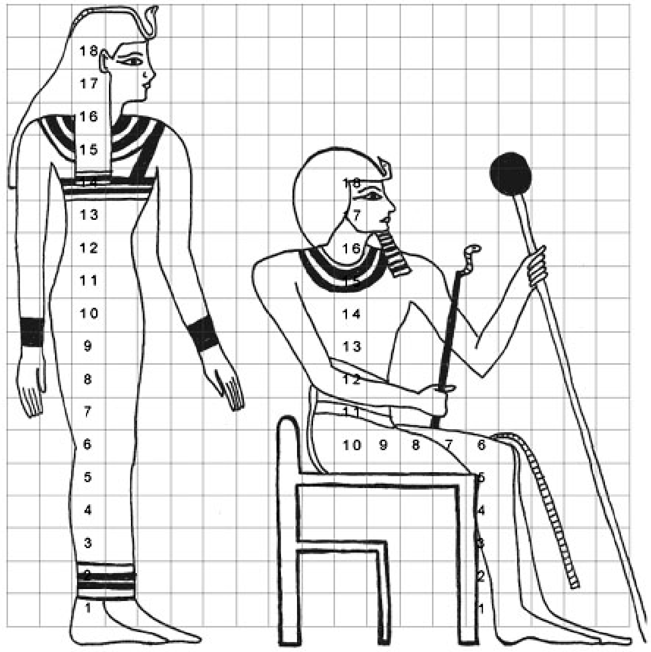


Fig. 6

There have been some doubts about whether working from a small sketch was a routine practice for the tomb painters because no such copybooks have survived. The small sketches, such as the ones on an ostrakon and white board might just as well have been a draftsman's practice piece or a model for an apprentice to copy.[37] But the important point underlying either interpretation is that space was imagined as flat and dissectible into squares, and the large and small world could be imagined as sharing the same structure. Any image could be scaled up, made larger, by increasing the size of the squares. The images would be identical because the proportions would remain the same, and the angular details of each square – smaller and larger – would correspond exactly. Insights into the big world could be accessible because the small world was *similar*; it shared the same structure.

What seems unambiguous is that the purpose of proportional rules for human figures was to render them in appropriate, recognizable forms. To do so, grids of different sizes were often prepared, but the proportional rules were followed.[38] Sometimes, a single large grid was placed over the tomb scene to accommodate the king, gods, vizier, or owner of a small tomb. Those figures were represented largest, and smaller figures, such as wives or workmen, were scaled-down; they were painted in smaller size but retaining comparable proportions to ensure that they too were immediately recognizable. Sometimes, however, different sized grids were used in the same scene or wall to represent the varying importance of the individuals depicted. Thus, the tomb painters imagined space as flat surfaces dissectible into squares, and the large world was projected as a scalable version of the small world. Since all squares divide into isosceles right triangles by their diagonals, *a fortiori*, Thales might well have come to a vision of the large world built up out of right triangles; the measurement of the pyramid height by its shadows focused on right triangles, as we shall explore shortly. Also, we see the same grid technique applied to sculpture. As Robins expressed it “There can be no doubt that Egyptian sculptors obtained acceptable proportions for their figures by drawing squared grids on the original block before carving it….”[39] When three-dimensional objects are also shown to be dissectible into squares, and thus triangles, a vision has been supplied of solid objects being folded up from flat surfaces. So, for a Thales measuring the height of a pyramid by its shadows at the time of day when every vertical object casts a shadow equal to its height, Thales had to imagine the pyramid and its measurement in terms of an isosceles right triangle, and the pyramid is easily imagined as four triangles on a flat surface the sides of which are folded up.

Let us think more about the geometrical diagrams Thales might have seen in Egypt. When Peet turns to examine the problem at MMP 6, the reckoning of a rectangle one of whose sides is ¾ that of the other, he insists that the solution is to be found “graphically,” although the papyrus does not contain such a diagram. He conjectures the following diagram:[40]

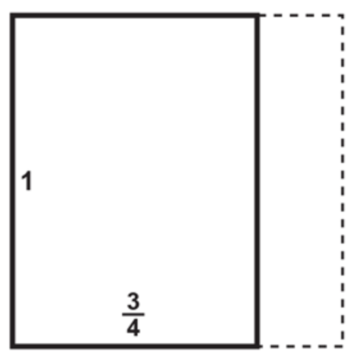
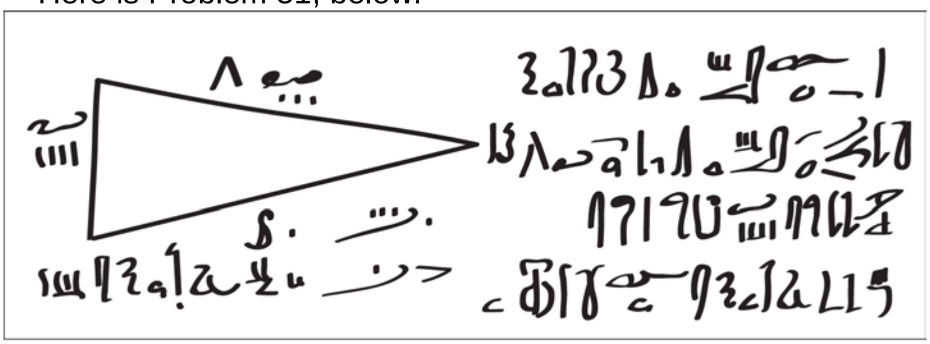


Fig. 7

First, the figure is imagined as a square, that is an equilateral rectangle, and then Problem 6 proceeds to compare the sides, reducing one side by 1⅓ to find the desired rectangle. The main points to emphasize for our purposes are that (i) the solution is imagined graphically, and (ii) the solution is reached through the construction of a diagram. The answer is found visually, and the uneven rectangle is reckoned by comparison with an even-sided rectangle, i.e., a square.

Thus, we have mathematical problems that concern reckoning pyramid height, the very problem with which Thales is credited with solving by measuring shadows and not by calculation. We should note that the pyramid problems in the RMP are, per force, problems involving triangulation. And the solution for these problems – calculating the inclination of the pyramid’s triangular face (i.e., *seked*) given the size of its base and height, or calculating the pyramid’s height given the size of the base and its *seked* -- the visual components are to set a triangle in comparison to its rectangle and square. In the RMP dealing with triangular areas [Problem 51] the triangle is placed on its side, and its "mouth" is the side opposite the sharpest angle. It seems as if here, as in other area problems, the diagram is "schematic"; just as in the medieval manuscripts of Euclid and other mathematics texts, the diagram is not meant to be a literal, specific rendition of the figure in the problem but only meant to suggest it more generally. Thus, historians of mathematics are forever debating the extent to which the figure matches the problem and whether or not this is a special case or just represents a range of cases, as here with triangles. Despite the earlier controversies, there is now general consensus that what had been a controversy concerning the formula for the area of a triangle has now been laid to rest: ½ base x height. Problems 49, 51, and 52 deal with the areas of rectangular and triangular pieces of land. Robins and Shute point out that the rough sketches have the triangles on their sides, not upright, as would be more familiar to us today.[41] Here is Problem 51, below:



After RMP Problem #51 in the British Museum

Fig. 8

Problem 51:[42]

> "Example of reckoning a triangle of land. If it is said to thee, A triangle of 10 khet in its height and 4 khet in its base. What is its acreage?
>
> The doing as it occurs: You are to take half of 4, namely 2, ***in order to give its rectangle***. You are to multiply 10 by 2. This is its acreage.

1	400	1	1000
½	200	2	2000

> Its acreage is 2."

Here '2' must mean 2 thousands-of-land. [We should have expected 20 khet, but the answer '2' is in the unit of "thousands-of-land" which equals 20 khet.][43]] The striking phrase, as Peet points out, is "*This is its rectangle*" [(*r rdj.t jfd.s*)]. Imhausen translates the sentence differently (into German) "*um zu veranlassen, dass es ein halbes Rechteck ist*." ["Then you calculate the half of 4 as 2] so that it is half a rectangle."[44] Clagett translates "Take ½ of 4, namely, 2, in order to get [one side of] its [equivalent] rectangle."[45] The triangle's area is grasped in the context of its rectangle with which it is inseparable. Peet argues that this problem, too, is meant to be solved graphically, and because it is not clear whether the diagram is schematic or a special case, he is caused to wonder how it might have been solved visually. Thus, he suggests a few possible diagrams of how it might have been imagined, but the main point for our purposes is to highlight that the triangle is grasped interdependently with its rectangle. The triangle was understood as a figure inseparable from the rectangle and the rectangle was understood inseparably from its square. And these interrelations of triangle, rectangle, and square show what Thales plausibly could have learned or confirmed in Egypt.

Peet's first conjecture about how to understand the problem visually is to try to sort out the technical terms in hieroglyphics to reach the view that this, below, is the likely diagram of the problem.[46]

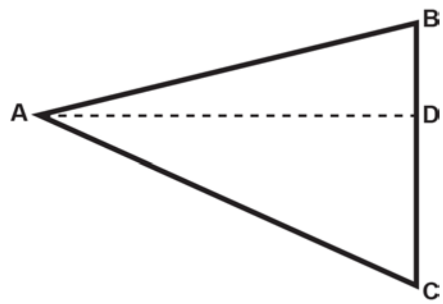


Fig. 9

And then, since the problem explicitly identifies the triangle in the context of a rectangle – Problem 51 includes the striking phrase: "This is its rectangle" – Peet speculates how the triangle might have been presented visually in the context of that rectangle, below.[47]

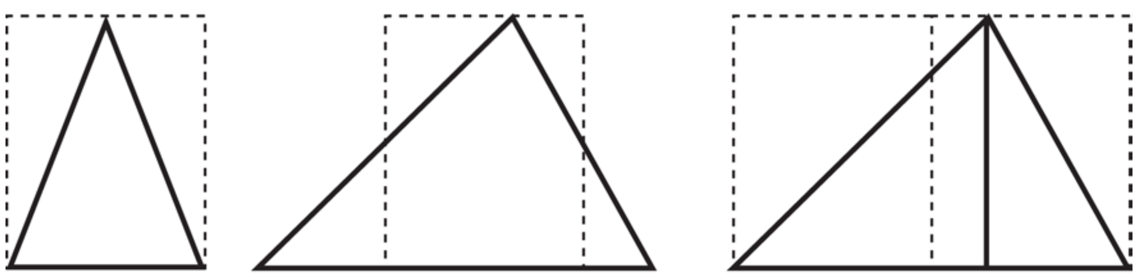
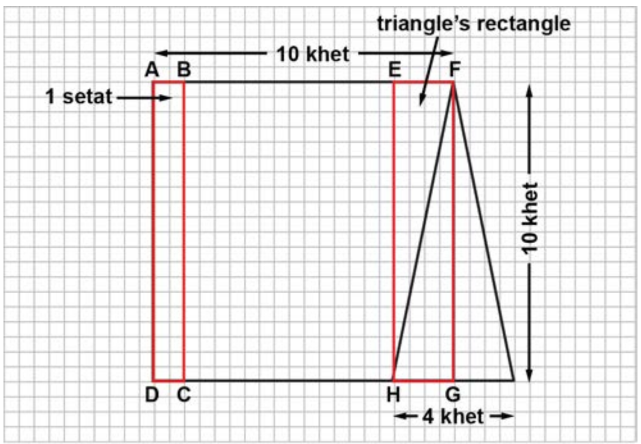


Fig. 10

Acknowledging that all this is speculation, let us consider another way that it might have been presented diagrammatically, below.[48] Since the preparation of a square grid was familiar in tomb painting and sculpture, let us place the diagram on a grid to explore its visual solution. We can recall the famous story told by Diodorus of the two Samian sculptors of the 6th century B.C.E. each of whom made half of a statue in different geographical locations – one in Egypt and the other in Samos -- following the Egyptian grid system; when the two halves were finally brought together they lined up perfectly.[49] So, since in both tomb painting and sculpture, the grid system was employed, and its knowledge is confirmed by the Greeks from Samos contemporaneously with Thales, I begin with the square grid for this diagram. And since it should now be clear that the relations between triangle, rectangle, and square were inextricably interwoven, I follow through, as does Peet, with some efforts to represent these problems for the visual solutions that he recognizes were likely *de rigeur*. The whole square of 10 khet is equal to 100 setat, each made up of 100 cubits-of-land, and thus each strip 1 khet wide and 10 khets long is therefore 10 setat of 100 cubits-of-land, and thus 1 strip is 1 thousand-of-land [1 = 1000], and 2 strips is 2 thousands-of-land [2 = 2000]. Thus, the correct answer is the *triangle's rectangle,* that is, 2. And visually, the answer is immediately apparent.



Rectangle ABCD = 1 thousand-of-land (= 1 setat)
Rectangle EFGH = 2 thousands-of-land (= 2 setat)

Fig. 11

In Problem 52, the challenge is to reckon the area of a truncated triangle of land, or trapezium:[50]

> "Example of reckoning a truncated triangle of land. If it is said to thee, A truncated triangle of land of 20 khet in its height, 6 khet in its base and 4 khet in its cut side. What is its acreage.
>
> You are to combine its base with the cut side: result 10. You are to take a half of 10, namely 5, ***in order to give its rectangle***. You are to multiply 20 five times, result 10 (sic). This is its area.
>
> The doing as it occurs:

1	1000	1	2000
½	500	2	4000
		4	3000
			Total: 10,000
		Making in land 20 (read 10)	

> This is its area in land."

Again, while Peet translates "You are to take a half of 10, namely 5, ***in order to give its rectangle,***" Clagett translates "Take ½ of 10 i.e., 5, ***in order to get [one side of] its [equivalent] rectangle***."[51] Regardless of the translation it seems clear that the triangle is understood to be connected inextricably with its rectangle of which it is half. The rectangle dissects into triangles; the triangle is the building block of the rectangle. Peet also conjectured that this problem, too, was to be solved graphically.

I emphasize again that these problems show how *land* in triangular and truncated triangular shapes could be reckoned. These show plausibly the conceptual principles behind the practical work of surveyors who sometimes, after the annual flood, had to return to the man who worked the land a plot of equal area but in a different shape. Guiding the calculation of areas of triangular and truncated triangular plots shows the kinds of flexibility needed since all rectangles reduce ultimately to the summation of triangles.

Thus, the Egyptian mathematical formulas practically applied in everyday life, polygon triangulation of land re-surveying, geometrical similarity revealed by the grid technique of tomb painting and sculpting, and the inextricable connection of triangle, rectangle, and square, were the kinds of things that likely inspired, educated, challenged, or possibly just confirmed what Thales knew. The resolution of the problems visually requires us to imagine the routine production of diagrams. These are parts of Herodotus' report that the Greeks learned geometry from Egypt and that, according to Eudemus, Thales introduced geometry into Greece having traveled to Egypt. Minimally, there were some geometrical diagrams in Egypt for which we have evidence. There can be no doubt, there were so many more, though about how they may have looked, we can only conjecture. But certainly, here was one resource that plausibly inspired Thales' diagram-making.

The metaphysical doctrine is that appearances are deceiving. Despite the fact that the many things we hear, touch, see, taste, and smell seem different, they really are, at base, the *same*. Although our experience is filled with different appearances – some fiery, airy, watery, and earthly – these appearances are one and all only this single underlying unity somehow modified to seem different. Had Thales announced this metaphysical insight, surely someone in his retinue would have asked "How does this happen?" How does a single unity -- ὕδωρ -- transform to appear fiery at one moment and at another liquid and flowing, one moment heavy and hard as stone and at another moment light as air? The answer to this unavoidable question had two parts: (i) there must be a basic building block out of which all appearances are compounded, and geometry offered a way to investigate and identify this fundamental figure, and (ii) there must be some process by which these modifications are produced; the process turned out to be *compression*, anticipating his younger contemporary Anaximenes' explication of rarefaction and condensation, and which I have discussed elsewhere.[52] In this essay I will address only the first point, how the search for the basic building block brought together philosophical investigations into nature, on the one hand, and geometrical inquiries, on the other.

E

Greek Geometrical Diagrams in 7th and 6th centuries B.C.E.

I now turn to consider that practical diagrams were familiar to the Greek architects/engineers and at least some of those working at monumental temple building sites long before the middle of the fifth century. Though the evidence that survives is meager, it provides an enormous window into the use of practical diagrams. When one sees that there is evidence in Greece already from the middle of the 7th century showing geometrical diagrams instructive for architectural purposes, and that lettered geometrical diagrams can reasonably be inferred as early as the middle of the 6th century, then, since Callimachus attributes to Thales the making of geometrical diagrams, and Eudemus testifies to them indirectly by attributing theorems to him, there can be no good grounds to deny that Thales was making geometrical diagrams. To see Thales making geometrical diagrams, we must set him, at least in the very beginning of his reflections on geometrical matters, in the context of those making practical diagrams. Moreover, having placed Thales in Egypt to measure the height of a pyramid, their long established traditions could have shown him how diagrams were part of the discussion and solution of geometrical problems.

It had been thought, along the lines of Netz' approach, by those like Heisel[53] that there were no surviving architectural drawings before Hellenistic times. Clearly, he did not yet know the drawings from the Artemision, published by Ulrich Schädler.[54] Schädler found geometrical sketches on tiles that had been burned in a fire, and so preserved, from the roof of an early temple of Artemis in Ephesus. The tiles date to the middle of the 7th century B.C.E., that is, before the time of Thales' flourishing. The drawings had been incised into wet clay.[55]

The concentric circles on the tile, below right, are a painted decoration of the edge of the roof. These tiles were made in an L-shape (turn the L 90° to the right) so that the front hangs down from the roof to protect the wooden structures. Its outer face was fashioned like a row of leaves with a not quite semicircular lower edge.[56] These concentric circles decorated the front.

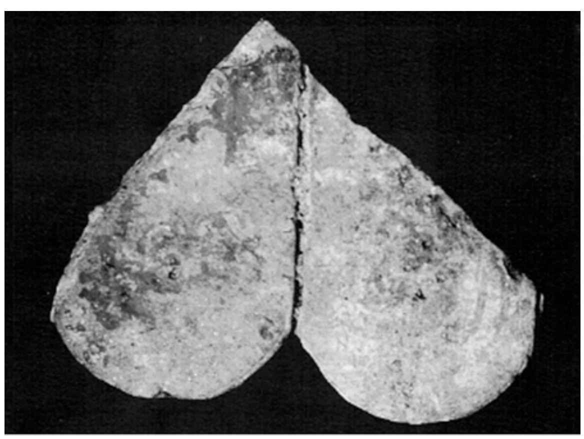


Fig. 12

Schädler shows the details in his reconstruction illustration, below. The tiles are of normal size (as far as we can tell, since they are fragmented, but the thickness is the same). The drawing is also of 1:1 size of the later ornament and not scaled down, as can be seen from the superposition of drawing and ornament, below.

Clearly, the drawing was not to be seen (why should it?). Schädler's claim is that it was a drawing of a working process, helpful for the workmen. What was to be seen was the ornamented border of the roof with the row of leaves and the concentric circles on them:[57]

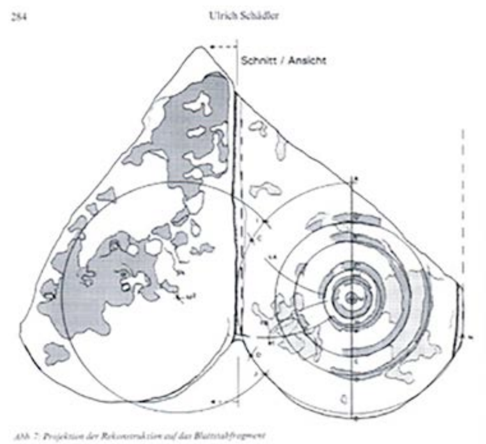


Fig. 13

The geometrical sketch on the slab, below, is a drawing made with the purpose to define 1) the curve of that lower edge of the roof tile and so the width of the "leaves," and 2) the diameters of the concentric circles, all mathematically quite exact, although the geometric method is just an approximation, but close enough to the mathematical proportions. The radius of the circle is 5.2cm = 1/10 of the Ionic cubit. Thus, the circles in the drawing (that have been defined by the sketch) match the circles on the tile, and the point L defines the width of the "leaf," i.e. the point where two leaves join and where the lower curved edge of the tile ends. Schädler's claim is that with the help of the sketch, the architect or workman defined a) the diameters/radii of the concentric circles and b) the lower edge and width of the leaves, and all this on the basis (5.2cm) of the Ionic cubit.

Fig. 14

In his reconstruction drawing, Schädler clarifies the details on the tile, below:

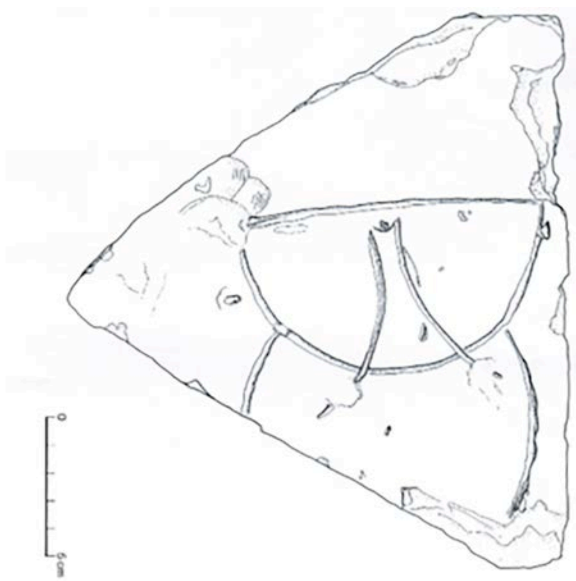


Fig. 15

Because these are among the only tiles showing geometric sketches that have survived from monumental Ionic architecture of the 7th century– they were preserved as a result of a fire and clearly were not intended as a permanent record of the technique -- they must be regarded as extra-ordinary, but they prove that already by this time the architects or workmen used geometrical methods to define measurements and proportions of certain details of their buildings. When they did so for the ornaments of the roof, it must be regarded as an even greater likelihood, despite the lack of surviving evidence, that they certainly did so for the broader outlines too, and other architectural elements. And what all this shows is that even at this early stage the measurements and proportions of the temples were based on numbers, mathematically exact, not just casually selected or random. To begin to understand Thales' diagram making, we must place him working within this context of practical diagrams.

Now I turn to the tunnel of Eupalinos whose excavation is may be placed in the middle of the 6th century, circa 550 B.C.E., that is, earlier than the time of Polycrates.[58] The tunnel is driven from two sides separately as a time-saving strategy; Herodotus refers to it as "double-mouthed" [ἀμφίστομον].[59] I discuss the construction of the tunnel in relevant details in Chapter 3 of *The Metaphysics of the Pythagorean Theorem*, but here my discussion is directed specifically to the argument that Eupalinos had to have made a scaled-measured diagram to achieve his successful results of the detour in the north tunnel– which is the excavator's claim: “There can be no doubt that an exact architectural survey of both tunnels had been established before the commencement of the correction. There can be no doubt either that the further digging was planned according to these plans so that the puncture could be successful with as little effort as possible and the highest certainty possible.”[60] The case for such a plan is an inference from a study of the triangular detour in the north tunnel and the measure marks painted on the western wall of the tunnel itself. We have no surviving plan but we do have the lettering on the western wall of the tunnel, and the inescapable picture that forms is one that points to a *lettered-diagram* – a scaled-drawing -- that was in turn transferred to the tunnel wall. Here is the argument.

To drive the tunnel from two sides separately, since they meet directly under the smooth ridge of Mount Castro, the crest of which is enclosed within the city’s fortified walls, Eupalinos had to have a clear idea from the start of the lengths of both tunnels, south and north. To do so, he staked out the hill and determined that both proposed entrances were fixed on the same level. This means, he set up stakes in straight lines – gnomons – running down from the summit of the hill, on both the north and south sides of the hill. Below, from Kienast, we have the white dots indicate where the stakes might have been placed down and along the hillside, from the top of the ridge of Mount Castro to the entrance of the south tunnel (= the lowest dot), all of which are within the city’s fortified walls, below:

Fig. 16

If we imagine that each white dot is a stake, Eupalinos had to measure carefully the horizontal distances between all the stakes on each side of the hill. By summing up all the horizontal distances, he had his hypothesis for the length of each tunnel half. In measures of our modern meter-lengths, the south tunnel was approximately 420 m, while the longer north tunnel was 616 m. In any case, the tunnels were not of equal length, and the fact that they meet under the ridge is evidence that supports the hypothesis that the hill was staked-out: this is *why* the tunnels meet directly under the ridge, and not midway between the two entrances. The simple geometry of this reasoning is presented, below; the addition of the green horizontal-lengths on the right side, and the blue horizontal-lengths on the left side, summed to the length of the tunnels.

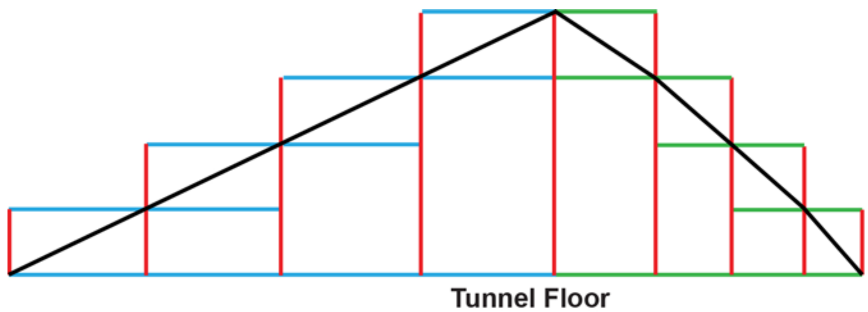


Fig. 17

Eupalinos' task was to bring the straight line, up and along the outside of the hill, into the hill itself as the tunnel line. As the digging proceeded, so long as he could turn back and still see the light from outside, he could be sure he was on track, or at least working in a straight line.

But in the north tunnel, after about 200 m, the rock began to crumble and Eupalinos feared the likelihood of collapse. Kienast's analysis of the course of the tunnel shows that the detour took the form of an isosceles triangle. It is the planning for and controlling of this triangular detour that required the scaled-measured diagram. The strong evidence for the existence of a scaled-measured plan – and hence a lettered diagram -- is the transposition and continuation of the Milesian system of numbers painted in red on the western wall of the triangular detour. The letters on the tunnel wall suggest at least an informal sketch with numbers or letters attached that was the reference for painted letters on the wall as the digging progressed, to keep track of the progress; but the need for a very accurate diagram, and hence the plausible case for it, came when Eupalinos was forced to leave the straight line in the north tunnel.

In the south tunnel, the uninterrupted letters, painted in red on the western wall, show us the marking system of lengths. The letters, from the Milesian system of counting numbers -- I, K, Λ, M, N, Ξ, O, Π, Ϙ (10 through 90) -- starting with I = 10, K= 20, Λ, = 30 and so on, run in tens to a hundred, P. The measure marks begin from the exact place at which the digging started and continue without interruption through the south tunnel. This was what Kienast refers to as System 1 - the original marking system for tunnel length, painted on the western wall. Since the measure marks are at intervals of every 20.60m, and are in units of ten, Eupalinos selected as his module 2.06m, and divided the whole tunnel length into some 50 parts, for a sum-total of approximately 1036 meters. And since this length of 2.06m does not correspond to any other basic unit of foot or ell – Samian, Pheidonic, Milesian[61] – Kienast suggested that it seemed reasonable to claim that Eupalinos invented his own tunnel measure, his own tunnel module.[62] In the context of early Greek philosophy, here was another example of the *One over Many*, just like the other contemporaneous standardizations of weights and measures, and of course, coinage. The architects of the Ionic stone temples in the 6th century -- the Samian Heraion, Dipteros I, c. 575, the archaic Artemision c. 560, and the Didymaion of Apollo c. 560 -- also needed a module to plan their gigantic buildings, so that the architectural elements would display the proportions they imagined and thus the finished appearance they intended, and accordingly each was reckoned as a multiple or submultiple of this module. Vitruvius tells us that their module was *column-diameter*.[63] When Anaximander expresses the size of the cosmos in earthly diameters that he described as analogous in shape and size to a 3 x 1 column-drum – 9+1 earth diameters to the wheel of stars (9/10), +9 to the moon wheel (18/19), + 9 to the sun wheel (27/28) -- he is not only making use contemporaneously of a modular technique but moreover has adopted exactly the architects' module – column diameter – to measure the cosmos. It is in this sense that Anaximander came to imagine the cosmos by architectural techniques because he came to grasp the cosmos as built architecture, an architecture built in stages like the great temples.[64] The fact that Anaximander wrote the first philosophical book in prose at precisely the same time that the Samian and Ephesian architects wrote prose treatises is no mere coincidence. It suggests the overlap and interaction of these two communities of interests.[65]

Thus, in the south tunnel the Milesian letters painted in red on the western wall allowed Eupalinos to keep track of his progress. The original plan is preserved, then, through the spacing of the letters on the wall, and suggests that there was an original diagram consisting of two straight lines – north tunnel and south tunnel – that met at distances directly under the ridge, and each tunnel length was expressed in units of 10 of Eupalinos' tunnel modules of 2.06 m, following the Milesian system of letters: beginning with the Greek letter iota [I = 10] and following the Greek alphabet to the letter rho [P = 100], after which the series begins again and continues to sigma [Σ = 200], and then beginning once more and ending in T [T = 300]. The letters painted on the western wall, then, correspond to the same letters on a diagram; thus, the existence of such a diagram is an inference from the lettering on the wall. This is not a matter of keeping track of a few marks; the tunnel is divided into some 50 parts of 20,60 m each and consequently required dozens of letters painted on the western wall to follow the progress of tunnel length.[66] This whole matter of how Eupalinos worked is both complicated and clarified by and through problems he encountered in the north end. When Kienast began to study the measure marks in the north end, he first discovered that they had been *shifted inward* on the triangular detour. When and why?

We now understand that as Eupalinos supervised the digging in the north end, he kept track of the progress in carrying out his plan by measure marks, just as he did in the south tunnel. He knew by his hypothesis the total length of the north tunnel by counting up the horizontal distances between the stakes, and expressed in the system of Milesian numeration, followed the progress as the digging proceeded. In the north tunnel only one letter from the original series is still visible on the western wall, the letter Λ. All the other letters of the original marking system are now hidden behind strengthening walls that were constructed *after* the completion of the tunnel to insure that the walls remained stable. But, according to Kienast, the Λ is at exactly the right place to correspond to the original marking system that starts, just as in the south end, exactly where the digging began. It was by means of these lettered marks that Eupalinos checked the progress in the tunnel excavation. Then, after about 200m, the rock began to crumble and Eupalinos feared collapse. And he was now faced with the problem of figuring out how to make a detour and yet still arrive at the originally planned meeting place directly under the ridge. When Kienast examined the triangular detour that comes next, he discovered that the letters are no longer in the places they would be expected. Instead, these measure marks appear "shifted inwards" towards the point or re-joining the straight line; they are in the same uniform distances, but they had shifted? How shall we account for this?

Kienast's theory is that Eupalinos had to abandon the straight line. Since the natural contour of the stratigraphy folded to the northeast, Eupalinos dug away from the crumbling stone towards the west. Kienast's theory is that, before the detour was begun, Eupalinos made a careful inspection of the terrain on top of the hill under which the detour would be needed, and noticing the conditions of the stone and stratigraphy imagined rather carefully exactly how long the tunnel would need to detour to the west before the tunnel line could be turned back to reach the original straight line. It is at just this moment, according to Kienast, that Eupalinos made a scaled-measured diagram, or added it to the original diagram of the tunnel line. The tunnel shows that Eupalinos planned the tunnel detour in the form of an isosceles triangle. He would dig westward at a specific angle – roughly 22-degrees – and when the stone again became safe, he would turn back at exactly the same angle.

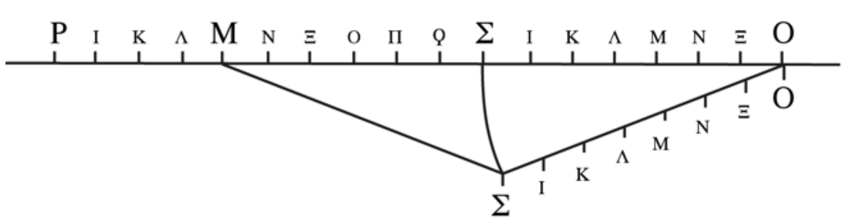


Fig. 18

Now, here is where we have the evidence for Eupalinos' use of a scaled-measured diagram. He knew the original straight line, and hence the originally calculated distance to dig from 'M' to 'O'. But, having left the straight line at 'M', he placed his compass point on 'O', on the originally planned straight line, and swung an arc that began from Σ – almost the midway point between M and O – and intersected the vertex of his proposed triangular detour and marked it Σ on his scaled-measured diagram and later on the tunnel wall itself at the vertex of the triangular detour when he reached this point. Kienast's argument is that the painting of Σ at the vertex *proves* that Σ was intentionally part of the design plan. Now, since Σ on both the original diagram and on the triangular detour in the tunnel itself were both radii of the same circle (i.e. arc), the length left to dig from Σ to I, K, Λ, M, N, Ξ, O (O is on the originally planned straight line) he already knew by the original plan. That sequence was followed on the second leg of the triangle from the vertex at Σ to O. All those letters are in the original intervals of length and painted on the second leg of the triangular detour.

The unresolved problem with this proposal is that the detour caused the total length of tunnel to be prolonged, and Eupalinos needed to know – as he made his scaled-measured diagram -- exactly how much extra digging was required to complete this detour, and so, he began to *count backwards*, and *mark backwards on the diagram* the intervals (i.e., same lengths), from the vertex of the detour, Σ, to the original straight line, now on the first leg of the triangular detour. The result was to shift the measure marks inwards, towards rejoining the original straight line – this Kienast explains as System 2 or *new marking system*, shifting inward as a result of the triangular vertex at Σ [Ϙ, Π, O, Ξ, N, M...Λ, K, I...and so on].

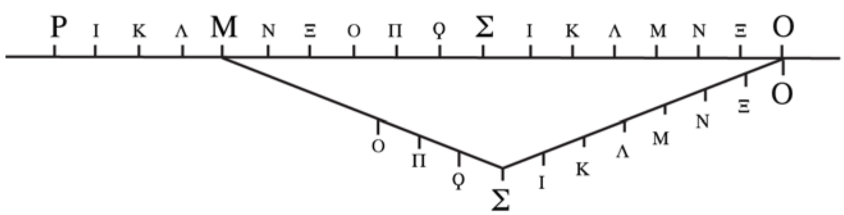


Fig. 19

Counting back to reach 'M', on the diagram, the distance lengthened by the detour would have become apparent, without any complicated arithmetical computation, below. The placement of 'M' on the first triangle leg, below, and on the tunnel wall itself, is not where the M should have appeared, as it does on the original straight line. This distance between the M on the first triangle leg and its original positioning is exactly the prolongation of the tunnel as a result of the detour; the additional distance from the "shifted M" to the M on the original straight line is a distance of 17,59 m.

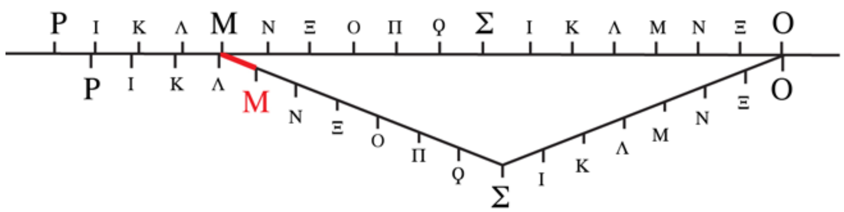


Fig. 20

Thus, the planned detour took the shape of a triangle; by digging westward, and then back eastward at the same angle, he was imagining the detour as an isosceles triangle. Both angles turned out to be roughly 22 degrees; and if the base angles of an isosceles triangle are equal then the sides opposite them must also be equal in length – this is one of the theorems attributed to Thales.

Kienast's thesis, then, is this: Since there is the letter Σ painted in red on the west wall of this detour at the vertex of this triangle, along with other Milesian letters in the series, Eupalinos *must* have made a scaled-measured diagram with the original straight line marked with Milesian letters for increments of 10 modules each, along with the proposed detour. With the aid of a compass whose point was set on the original straight line 'O', he drew an arc from the Σ on the original straight line to the vertex of the triangle-detour. Since both segments OΣ on the diagram – the original segment on the straight line <u>and</u> the segment on the second side of the triangle detour -- are radii of the same circle (of which this is an arc), the length of both segments must be the same. Then, as work proceeded to rejoin the original straight line at O in the second side of the triangle detour, the same distances were transposed to the west wall of the second leg of the triangle to follow the progress I, K, Λ, M, N, Ξ, O. By the scaled-measured diagram – *before* the digging of the detour started -- the first leg of the triangle could be marked off in lettered distances – in reverse -- Σ, Ϙ, Π, O, Ξ, N, M, and the lengthened distance caused by the detour would become immediately apparent – 17.59 m without the added complexity of arithmetical computation. Thus, by means of the scaled-measured plan – the lettered diagram – he knew *before* he started the detour a very close approximation of the extra digging required to complete the tunnel successfully.

And finally, there is written on one of the strengthening walls – which means it was added *after* the tunnel was finished -- the word PARADEGMA; it has painted on both sides, left and right, a vertical measure mark:

Fig. 21

The remarkable thing is that the distance from the left vertical measure mark to the right is almost exactly 17.59 m,[67] and for this reason Kienast, following the suggestion of Käppel, embraced the view that Eupalinos celebrated his calculation of the prolongation of tunnel.[68] Since PARADEGMA is painted on a strengthening wall, it was clearly painted *after* the work was completed and so could not have served a purpose in the process of digging the tunnel.[69]

F
Thales' Diagrams in Context

Now, having set out a broad context of Mesopotamian, Egyptian, and Greek geometrical diagrams I propose to construct a plausible case that Thales had taken from the Egyptians, or interpreted in his own way as a product of their preoccupation with land surveying following the annual inundation, that all space was imagined as compounded out of flat surfaces, structured by countless straight lines, and articulated by innumerable rectilinear figures, the volumes of which were folded up or layered up. All rectilinear figures reduce to triangles to determine their areas. This is why, I contend, that the RMP #51 offers instructions on calculating the area of a *triangular plot of land* – not a triangle, but a *plot of land*. Herodotus says that the Pharaoh divided the land into squares but the evidence suggests that this was mistaken, and instead the land was divided into rectangles, that is, right-angled figures with unequal sides. When the Nile flooded, at times the turgid waters must have been so robust that the land was eviscerated, and along the banks sometimes simply washed away. In order to return to each man who worked the land a parcel of the *same area* – in order that a uniform tribute could be assessed as in the past – sometimes the parcel of land could not be returned in the same shape, but instead some other rectilinear figure of the *same area*. It was in these sorts of countless exercises in surveying that an ability to see equality of areas among rectilinear shapes was fostered. When is a rectangle equal in area to a square? When is a triangle equal in area to a rectangle? To be sure that the resurveyed land retained the same area, the "proof" or "demonstration" was furnished by the lesson of RMP #51: the plot was reduced to triangles and the area of the land could be reckoned as the sum of the areas of the triangular divisions.

This idea is preserved in Euclid VI.20. The figure is a pentagon but what it represents is *any* polygon. The theorem immediately prior in Euclid is *Similar triangles are to one another in the duplicate ratio of corresponding sides.* At VI.20 the theorem is *Similar polygons are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding side*. The theorem shows that all rectilinear figures, no matter their size, divide into triangles, each of whose area is the sum of the triangles into which it divides. Moreover, there is a pattern of expansion or contraction such that as line length increases from one similar figure to another the area of that figure increases or diminishes in the duplicate ratio of the corresponding sides.

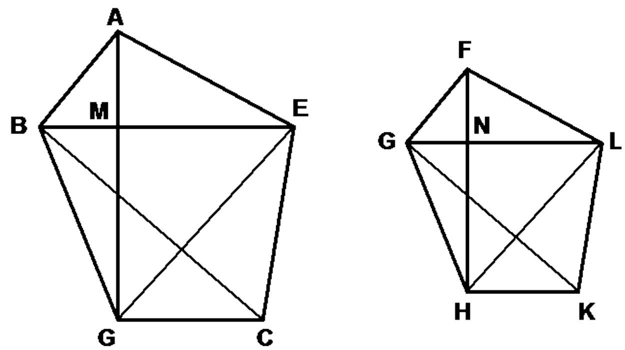


Fig. 22

Next, by exploring triangles, Thales was in a position to observe that all triangles divide into two right-angled triangles, and by continuing to divide them from the right angle, it becomes obvious that they divide indefinitely into two similar right triangles, as below:

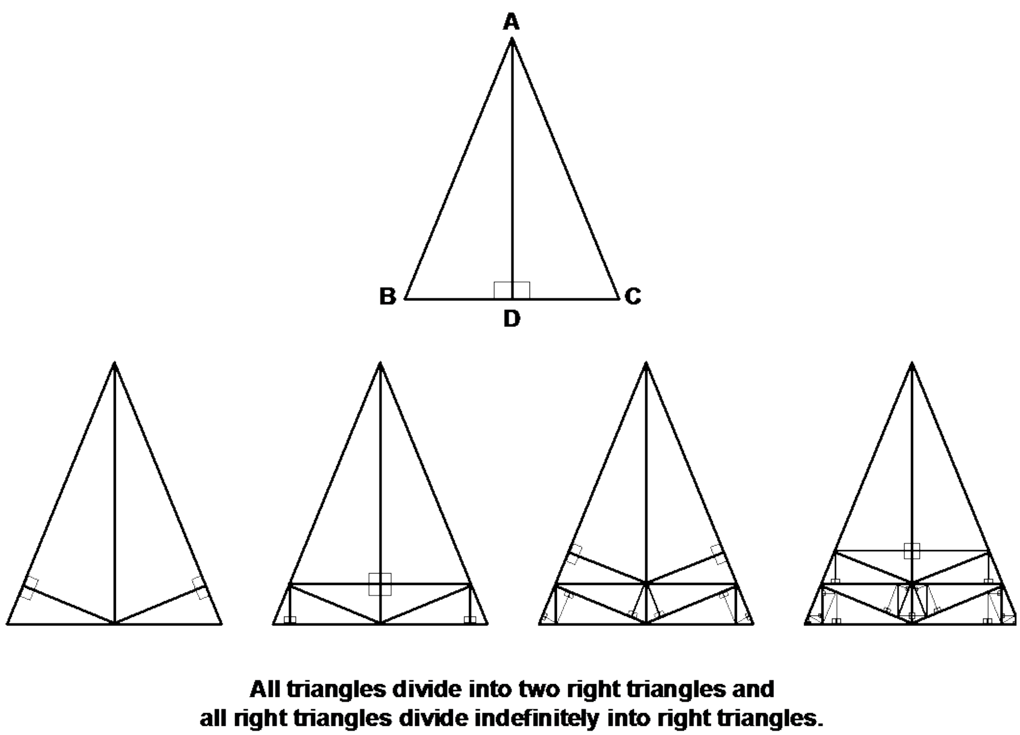


Fig. 23

With this background in mind, if one will continue to explore the diagrams that represent Thales' measurements of the pyramid and ship, and place them next to the diagrams for the theorems with which he is also associated, one can see that and why Thales explored triangles, since all rectilinear figures reduce to triangles. And as he investigated more, he realized that by dividing any triangle by dropping a perpendicular from the vertex to the side opposite, inside every triangle were two right triangles. And if he continued the process of dividing right triangles from the right angle, he discovered that inside every right triangle were two right triangles, each similar to each other and similar to the whole right triangle now divided…forever! This was not an atomic conception; there was no smallest right triangle but rather a pair of similar right triangles in endlessly diminishing size. And as he examined right triangles further, he discovered also that – in this process of producing smaller and smaller right triangles, or reversing that process, by expanding them into larger and larger right triangles – they collapsed and expanded according to a pattern. This pattern came to be known as a μέση ἀνάλογος -- "mean proportion" or "continuous proportion," or "geometric mean." And the visualization of this pattern consisted in showing that the square made on the perpendicular was equal to the rectangle made from the two line lengths into which the hypotenuse had been partitioned. To see this is to grasp the Pythagorean theorem, but not along the lines of Euclid I.47, credited by Proclus to Euclid himself, but rather along the lines of VI.31 – the so-called *enlargement* of the hypotenuse theorem -- the proof by ratios, proportions, and similarity. My contention is that Euclid VI perfects lines of thought first introduced by Thales and carried forward by Pythagoras and the Pythagoreans.

While the name "Pythagoras" has been popularly connected with the hypotenuse theorem, since the work by Burkert (1962/72),[70] an avalanche of scholarly opinion has discredited Pythagoras with this discovery, and more generally with any contribution to mathematics at all. Zhmud, more recently (2012),[71] has challenged afresh Burkert's positions, arguing for the plausibility of the connection of Pythagoras and theorem. But let us be clear that, with the exception of naming "Pythagoras," there is no other name from antiquity singled out for connection to this theorem -- no one. The earliest discovery of it *for the Greeks* and the authorship of stages for its demonstration – the Cuneiform tablets prove that it was known in some form by the Babylonians at least a millennium earlier -- are lost in the mist of ancient times. But if we will set out Thales' diagrams for the theorems with which his name is associated, and the diagrams presupposed by the measurements of pyramid and distance to a ship each of which relies on revealing features of right triangles and similar triangles, and place these investigations within the context of resolving a metaphysical problem of explaining *how* a single underlying unity can transform into the diversity of appearances, a plausible case that Thales knew an interpretation of the hypotenuse theorem can be produced. *What the hypotenuse theorem shows -- by ratios, proportions, and similar triangles – is that the right triangle is the fundamental geometrical figure into which all other rectilinear dissect*. I cannot find any indication in the scholarly literature that this point has been observed, nor almost none that connects "Thales" to this theorem. The exception is the work of mathematician Tobias Dantzig who suggested that Thales knew the hypotenuse theorem because it is a consequence of similarity, and that Thales grasped similar triangles.

E

THALES' DIAGRAMS

To have measured the height of a pyramid when the shadow was equal to height required imagining a right-angled triangle. The only way that Thales could have known that the time was right to measure the pyramid height equal to the shadow length was to have had a gnomon, or his own shadow, display the equivalence of shadow and height.

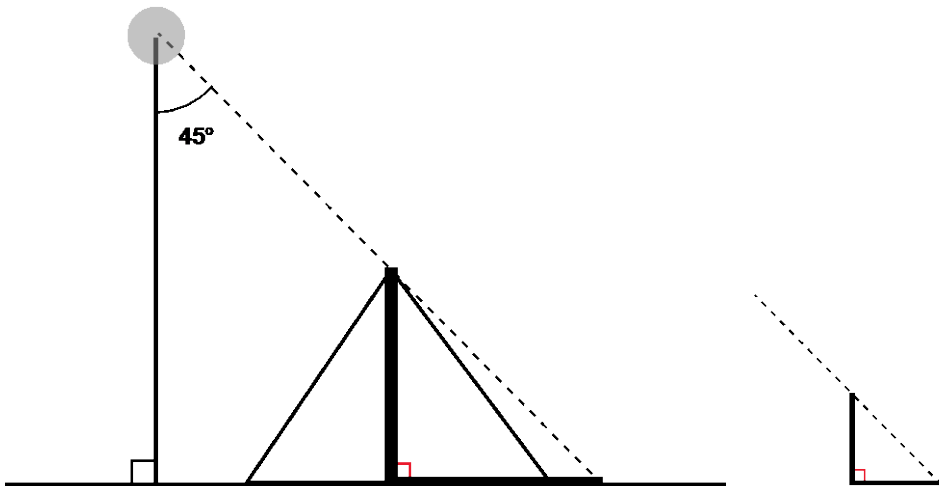


Fig. 24

To imagine the measurement when the shadow was un-equal but proportional to the height, again, required imagining a right-angled triangle and with a gnomon nearby or even at the end of the projected shadow to have the comparison from which he could confirm the proportional result. Per force, this, too, was an exercise in *similar triangles*.

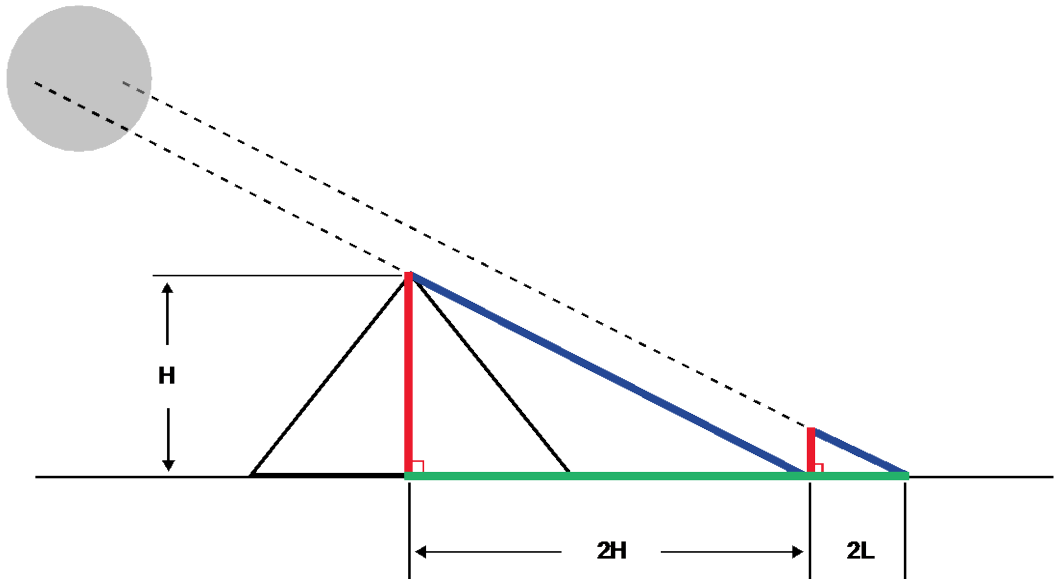


Fig. 25

Both measurements are exercises in similar triangles, the first is 1:1 similarity and the second extended the idea to countless possibilities; in both cases the angles in the triangle are equal but the lengths of the sides are not.

To have measured the distance of a ship at sea requires precisely this same imagination of right triangles, no matter how Thales actually carried it out, and probably from more than one point of view. He might have assembled his associates along the shore line AD holding a measured cord, below:

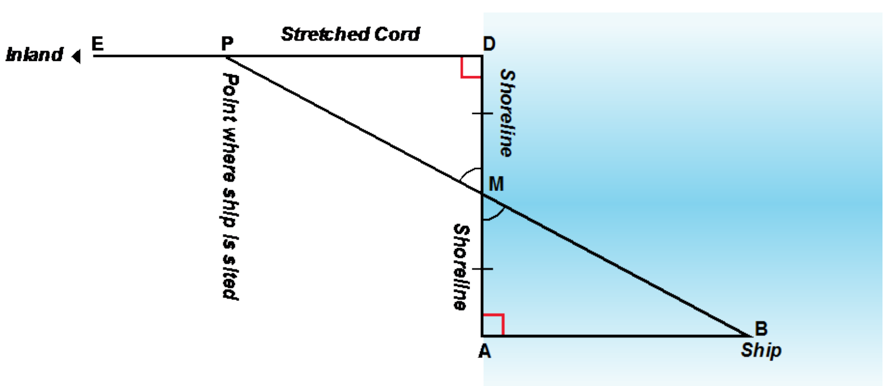


Fig. 26

Or, he might have made the measurement from a raised tower since the sea coast near Miletus has few flat plains.

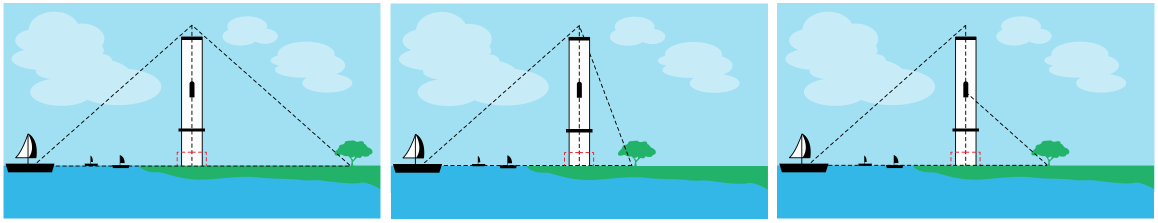
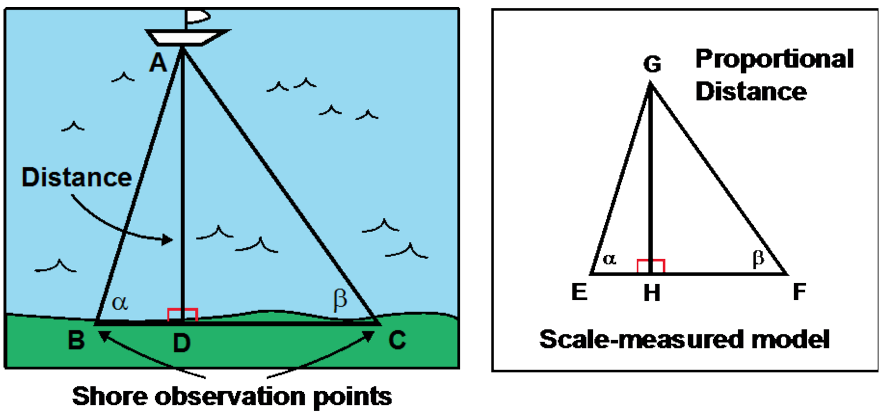


Fig. 27

Or, he might have reasoned proportionally from a scaled-measured diagram or proportionally-sized model, in a manner analogous to the way that Eupalinos may have reasoned in correcting the north tunnel detour:



The actual distance is calculated by proportionality. That is:

BC : AD :: EF : GH

Fig. 28

But, whichever way or ways he made this measurement, they all rested on grasping similar *right* triangles, that is, right triangles sharing the same angles but not the same line lengths, and thus sharing the same ratios and proportions.

Geminus' testifies in Eutocius' commentary on Euclid's *Conica*, that the Pythagoreans proved there were two right angles in every triangle. Here is the proof sequence, the last diagram (with angles supplied) is the one preserved by Proclus:

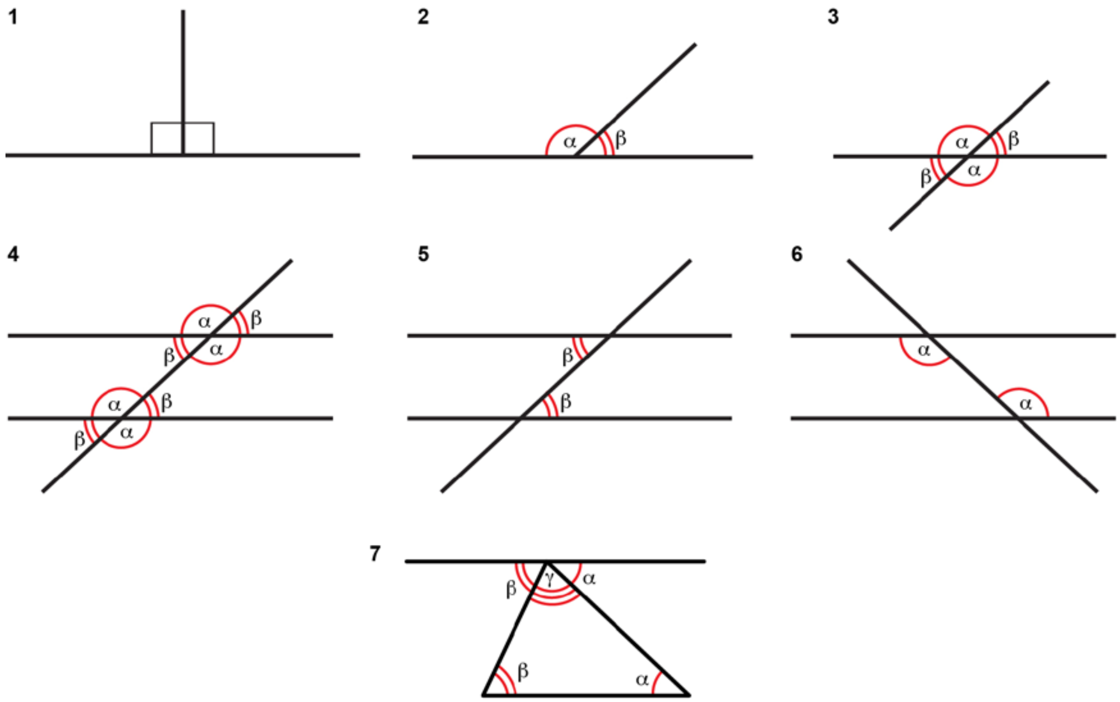


Fig. 29

Thus, (1) there are two right angles in each straight line, (2) no matter where you divide the straight line the two angles sum to two right angles, (3) opposite angles must be equal because any two of them sum to two right angles, (4, 5, 6) if you continue the diagonal straight line so that it falls on a line parallel to first, it is clear that the alternate and opposite angles β are equal, and the alternate and opposite angles α are equal. Finally, (7) if you add to the angles β and α, the angle γ, we see that α + β + γ sum to two right angles, and so the sum of the angles of any triangle sum to two right angles.

Now, Geminus continues, that while the Pythagoreans provided that proof, the "ancients" – who could be none other than Thales and his retinue – had investigated that there were two right angles in each species of triangles, equilateral, isosceles, and scalene. There are many ways that this could have been investigated but had he followed what he was in a position to learn from the Egyptian *RMP* #51 and 52, focusing on the *triangle's rectangle*, he would have noticed unmistakably that all species of triangles contain two right angles. Ordinarily, we tend to think of a rectangle, compared to a triangle, on the same base between the same two straight lines, as having twice the area of the triangle. This is likely because we imagine every rectangle divisible into two triangles by its diagonal. But as RMP #51 and 52 show, the area of a triangle is reckoned as a *triangle's rectangle*, and in this case, we are referring to a rectangle that is equal in area to triangle, and not double it. Had Thales constructed the triangle's rectangle around each species of triangle, he would have seen immediately that every species of triangle contained exactly two right angles, as below:

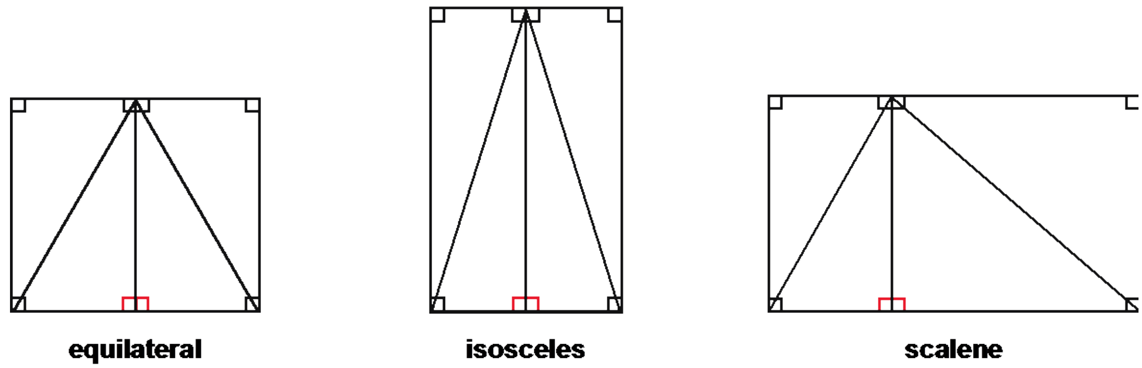


Fig. 30

Each half of the divided triangle contains two right angles because each is in its rectangle that contains four right angles. Since each side of the triangle divided its rectangle in half (i.e. was the diagonal in the rectangle), each triangle contains two right angles. And since the combined two triangles in their rectangles contain four right angles, if we remove the two right angles at the base, each triangle is revealed to contain two right angles.

Three of the other diagrams fit together in a manner that has remained unappreciated in the literature.

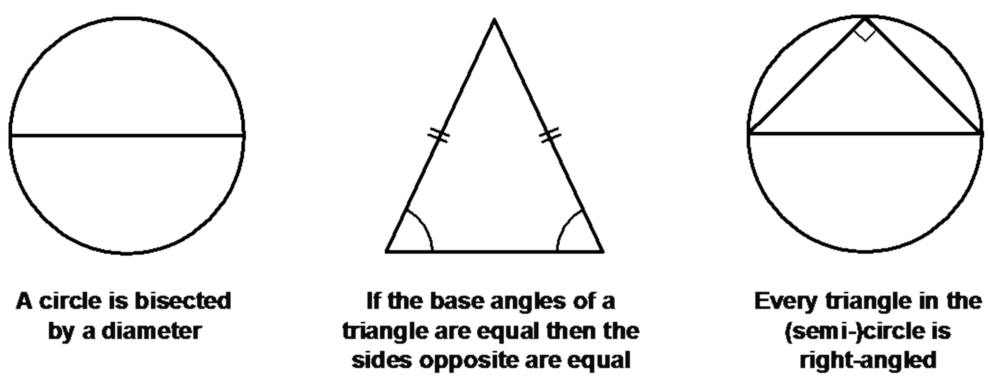


Fig. 31

Let us consider how these three theorems fit together: that a circle is bisected by a diameter, that if the base angles of a triangle are equal then the sides opposite are equal, and that every triangle in the (semi-)circle is right-angled. They all hold together as parts of the awareness that every triangle constructed in a circle on its diameter is right. In the past, scholars have suggested that while Thales' "proved" that the diameter divides the circle in half – supposing its construction was likely by simple superposition – Euclid relegates it merely to a definition, apparently not needing a proof at all. But, what I am suggesting is that this was the first step at getting to what Thales was really after, namely discovering for himself, and then showing others, that the right triangle was the basic building block of all other rectilinear figures, and *a fortiori*, space and all appearances in space itself.

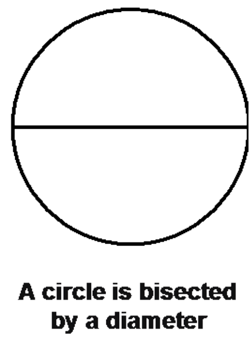


Fig. 32

Consider, next, the construction of any triangle in a circle where the two end points are the circles' diameter, where the (semi-)circle terminates at the diameter. It must be either isosceles (left, below) or scalene (right, below), and let us identify the angles that result.

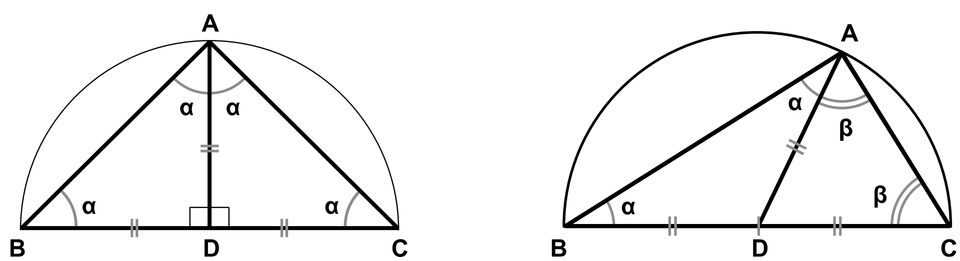


Fig. 33

Since Thales already knew, as Geminus reports, that there were two right angles in every species of triangle, he observes immediately, in the isosceles triangle in a (semi-) circle (left, above) that each angle 'α' is ½ right angle, and so at 'A' he sees immediately that the angle at A is 2α = 1 right angle. He knows this because the theorem he also knows is that if the two sides of a triangle are equal in length (they are all radii of the circle) then the angles opposite those equal sides must be equal. In the case of the scalene triangle, since he knows that 2α equals 1 right angle, and knows that sides DA and DC are also radii in a circle and so of equal length, he knows that the angles opposite them must also be equal; And since there are two right angles in each species of triangle, angles α + β must be equal to a right angle. At all events, it is my surmise that Thales began with the isosceles right triangle where areal equivalence is immediately obvious, as I am supposijng he did with the first pyramid measurement and the distances of a ship as sea, and then investigated the scalene triangle to see if these relations held there as well.

Diogenes Laertius preserves the report by Pamphile who claimed πρῶτον καταγράψαι κύκλου τὸ τρίγωνον ὀρθογώνιον, καὶ θῦσαι βοῦν that "[Thales was] the first to describe on a circle a triangle (which shall be) right-angled, and that he sacrificed an ox (on the strength of this discovery)."[72] Heath remarked that "This must apparently mean that Thales discovered the angle in a semi-circle is a right angle (Euclid III.31)."[73] As Heath sorts through the difficulties of these reports, he is almost able to see that, if we accept this attribution, Thales might well have discovered an areal interpretation of the hypotenuse theorem, and though neither Allman nor Gow attribute explicitly the discovery of the hypotenuse theorem to Thales (though Gow supposes that Thales may well have known the contents of the first six books of Euclid, and *a fortiori* must have then had knowledge of the hypotenuse theorem), they go much further to work out the connecting themes. It seems to me that Heath cannot see this because he never considers the metaphysical implications of these geometrical efforts; Heath seems to imagine Thales' preoccupation with practical problems, or with formulating proofs and demonstration as if he were in a mathematics seminar.

In addition, the testimony has been suspect because Thales' sacrificed an ox upon his discovery and this sounded much like the report that Pythagoras made a splendid sacrifice upon his discovery of the hypotenuse theorem. But let us for the moment entertain that these reports were no mis-reporting, nor a dittographical error, but rather are connected and connected to the same problem, the same geometrical intuition. What is it about this realization that merits a splendid sacrifice? I am in agreement with the surmise of Heath, and Allman and Gow before him, that the discovery is what later appears in Euclid at III.31. This cannot mean that he drew, or simply constructed, a right angled triangle on or in a circle because *every* triangle drawn in a (semi-)circle, whose ends are the points where the diameter meets the circle, and whose vertex is on the circumference, is right. Let us gaze upon the diagrams that capture this idea and follow the diagrams.

Allman conjectures that had Thales known that every triangle inscribed in a semi-circle is right, he deserves credit for the discovery of *geometric loci* – that if one plots all the points that are vertices of a right-angled triangle whose base is the diameter of a circle, all those points define the circumference of the circle. In this case, Allman is reflecting on some of the conditions that stand out when one inspects further this discovery – if Thales discovered that every triangle in a (semi-)circle is right, he would surely have realized also that the vertex of every right triangle in that (semi-)circle defines the circumference of the circle. I find Allman's speculations enticing but I would describe this insight differently, placing it in the context of metaphysical speculation rather than a mathematics seminar. When one discovers that every triangle inscribed in a (semi-)circle – whose vertex is at A and whose base is diameter BC – is right, one has discovered *all* the right-angled triangles that fill up and are contained within the circle. The circle is filled completely with and by right-angled triangles, the right angles at A mark out completely the circumference of any circle. Thales' investigations that led to this realization about right angles in the (semi-)circle came about as he was exploring what was the basic geometrical figure – circles, rectangles, triangles – and his conclusion, as we shall now examine further through VI.31 was that it was the right triangle. *Even the circle is constructed out of right triangles – that within every circle are all possible right-angle triangles!*

Thus ALL the possible right angles are potentially displayed in the half circle, and by placing in sequence all the right triangles, a circle is created by connecting their vertices at the right angle.

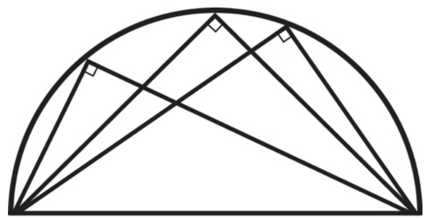


Fig. 34

And consequently, if ALL the right angles are continued in their mirroring opposite half of the (semi-)circle, then ALL possible rectangles are displayed in the circle, including of course, the equilateral rectangle or square.

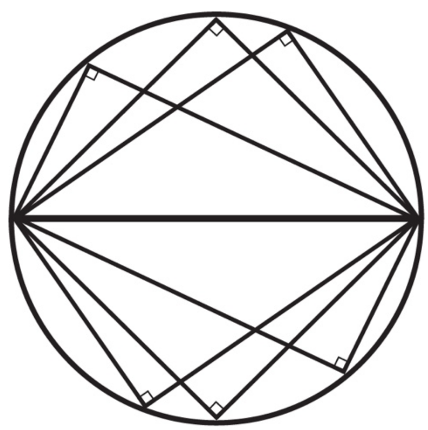


Fig. 35

And if one removes the semi-circle, one is left to gaze upon almost exactly the same diagram as VI.31 (without the figures drawn on the sides, and not necessarily having a radius but rather any line connecting the diameter to the semi-circle). In the ancient manuscripts, unlike Heiberg's later version, the diagram at VI.31 is an isosceles right triangle, containing no figures drawn on the sides [sc. the right angle box is supplied for clarity].

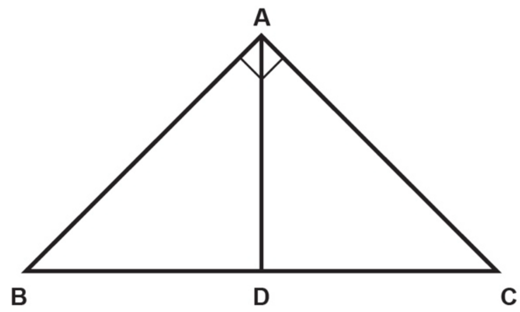


Fig. 36

The argument I am proposing -- that Thales plausibly grasped the hypotenuse theorem -- rests on the premise that this turned out to be the starting point to the answer he was seeking in his inquiry: How does a single underlying unity appear so divergently without changing but merely altering? It's not just that this theorem points unmistakably to the awareness that all rectilinear figures reduce to right triangles, but is a consequence of grasping geometrical similarity. First, then, we consider how VI.31 works in terms of similar figures, and then show that grasping what this meant, upon closer and focused attention, was that the right triangle expanded and collapsed in terms of a pattern of "mean proportion" because the parts of the right triangle formed a continuous proportion of lengths. This realization opened the project of trying to see how all appearances were constructed out of right triangles. The visualization of this relation was shown in terms of the relation of the figures that could be constructed on each of its parts, and almost certainly, at first, was explored in terms of squares and rectangles, though these relations hold for *any* figures similar and similarly drawn on the three sides including (but not limited to) equilateral triangles, semi-circles, and even irregular figures!

Let us consider, then, the very idea of geometrical similarity and see what it entails. I am arguing that the very idea of similarity dawns when one begins to see a structural connection between little and big things, especially when one is struck that big things have the same shape as little things – the big things are blown up or enlarged versions of the little things. The idea might very well first dawn if one is looking to see if there is some unifying connection between big and little things --- this recognition came to be developed in terms of what we call the "microcosmic-macrocosmic" relation. In the literature, we refer to a distinction between analogies and similes; but in both cases we draw attention to commonalities between two things despite the differences.

This same point can be made by looking at the archaic temples – the one planned and built to Apollo in Didyma for the Milesians, and of course, the temple to Artemis in Ephesus and the temple to Hera in Samos. In order to control the overall look of the gigantic temple so that it would have the proportions that were judged appropriate by those who planned for it, a module or basic unit was identified and then the rest of the architectural elements were reckoned as multiples or submultiples of it. While there has been much debate about which element was the module, there has been some consensus that it was reckoned in terms of lower column diameter. Seen in this way, the temple quite literally grew up and out of the fundamental module – a One over Many. When Anaximander identifies the shape and size of the Earth by analogy with a 3 x 1 column drum, and then reckons the size of the cosmos in "Earthly proportions" – that is, column drum Earthly proportions, the distance to the stars, moon, and sun as 9/10, + 18/19, + 27/28 column drum Earthly proportions – he is making use not only of a modular technique but specifically making use of the architect's module. To describe Anaximander's project in this way is to suggest that Anaximander imagined the cosmos in architectural terms because he came to grasp it as cosmic architecture, a structure built in stages just like the monumental temples.

Now, geometric similarity is a way of expressing a commonality between two things that share the same shape; similar figures have the same shape but are of different sizes. We might say that a bigger, similar right triangle (for example) is a scaled-up version of the smaller, just as a temple element is a multiple (= scaled-up) or submultiple (scaled-down) of the module. In the diagram, below left, right triangle A has three rectangles described on its sides and each is similar and similarly drawn. This means that each figure described on its sides is proportional. In the second diagram, below right, the same triangle is divided by a perpendicular from the right angle into two similar right triangles. This means that each of the three triangles has the same angles but whose sides are of different lengths. Moreover, the two triangles B and C sum to the large triangle A.

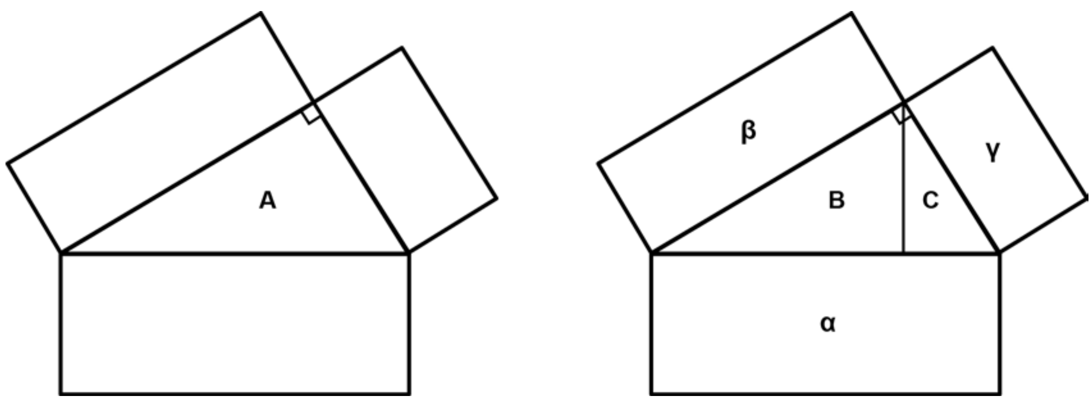


Fig. 37

But if one will inspect these triangles and rectangles further, one will discover that the ratio of the largest triangle A to the figure drawn on its hypotenuse, α, is the same ratio as the triangle B is to its figure β, and as triangle C is to γ. And this also implies that as β : α = B : A. Similarly, γ : α = C : A. But, A equals the sum of B and C, and thus α = the sum of β and γ.

Central to my argument is that I contend that Thales understood ***similar triangles***, and the Pythagorean theorem along the route to what is preserved at VI.31 is a consequence of thinking through right-triangle similarity. Every right triangle divided at the right angle by its perpendicular, contains two similar right triangles – each is similar to the other and similar to the whole. Each triangle shares the same angles but differ only with regard to the length of the sides.

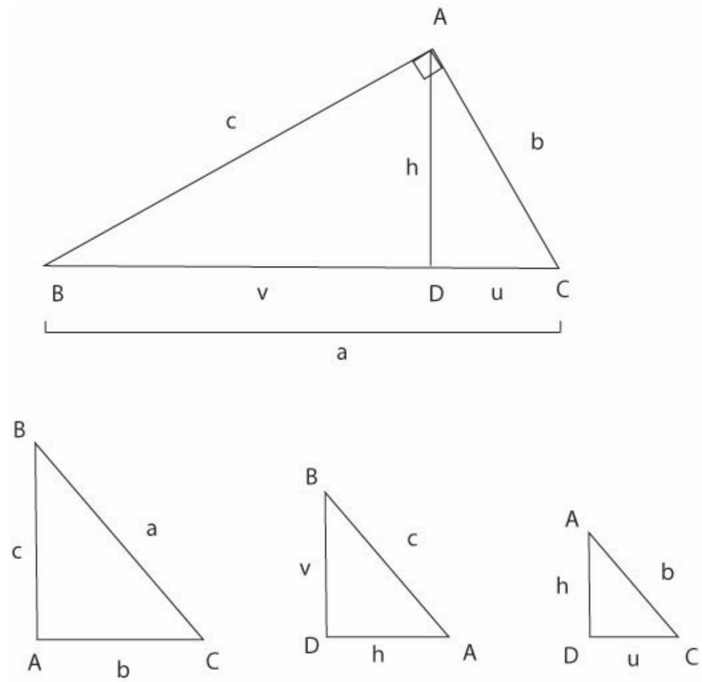


Fig. 38

This geometrical intuition of the interpretation of the Pythagorean theorem along the lines preserved in Euclid Book VI is the kind of inference plausibly attributed to Thales. Every rectilinear figure dissects into triangles, the sum of which is the area of any figure because the figure is built out of them. And every triangle dissects into two right triangles by dropping a perpendicular from its vertex to its base. And inside every right triangle are two right triangles, similar to each other and to the larger triangle so divided, *ad infinitum*. [N.B. there is no smallest right triangle; the conception is not atomic but a continuous, endless divide.] And each right triangle has the remarkable feature that it stands in relation to the two right triangles within it, so that the sum of the similar and similarly drawn figures on its sides sum to the figure on its hypotenuse, just as the two similar triangles within it sum to the larger triangle now divided. Because this relation holds without regard to the specific figures so drawn – they could be squares, rectangles, equilateral triangles, semi-circles, even irregular figures, but they must be proportionally drawn – this theorem at VI.31 in Euclid came to be known as the "enlargement" of the Pythagorean theorem. But in any case, a central point is that if one were to have the intuition that there was some shape that was fundamental, the shape into which all bigger things reduce, and consequently out of which the bigger things could be imagined as constructed, *this is the idea of similarity*.

Now let us put together the parts of Euclid's VI.31. We begin with a right triangle. From the diameter of the semi-circle we have a method that guarantees the triangle inscribed within it is right. And, as we have been discussing, we should see that the right triangle came to present a mystery. All figures reduce to triangles and all triangles reduce to right triangles. Did the right triangle divide further? How might Thales have explored this? Thales divided the right triangle by its perpendicular from the right angle. Was there a *pattern* by means of which the right triangle further dissects? And if so, how should this pattern be described? As he investigated further, he discovered that pattern; the parts -- perpendicular and two segments into which it divides the hypotenuse -- form a continuous proportion. Since he knew, from watching Egyptian surveyors work, that there was a correspondence between line lengths and the area of figures created by them, he made various squares and rectangles on the perpendicular, and the parts of the hypotenuse into which the perpendicular divided them. He used the line lengths that were already supplied and compared and contrasted them. What he discovered was that the perpendicular was the “middle term,” as it were, between the hypotenuse parts. He came to grasp that the perpendicular from the right angle was the *mean proportional* or *geometrical mean* between the two side lengths, and moreover, its corollary that the perpendicular divided the original triangle into two similar triangles, each of which shares a mean proportional with the largest triangle, Was there more to describe this pattern? He became aware that there was a correspondence between line lengths and the figures drawn on them; when line lengths stand in continuous proportions, the length of the first is to the length of the third in duplicate ratio as the area of the figure on the first is to the area of the similar and similarly drawn figure on the second. This is precisely what an understanding of *right-triangle similarity* implies. And all these details provided a way of expressing just what he was looking for, namely, *how* – the pattern -- the right triangle dissects indefinitely into right triangles, indeed collapses by mean proportions, and how the right triangle expands so as to produce areal equivalences on its sides again in mean proportions. This was the key to Thales’ grasp of a project to look for *transformational equivalences*, the project of the cosmos growing out of right triangles. Pythagoras is credited with the knowledge of <u>three</u> *means*: arithmetic, geometric, and harmonic, and if we accept this, it is certainly not too much of a stretch to realize that Thales understood the idea of a geometric mean contemporaneous and earlier. The geometric mean or mean proportional was the grasp of the continuous proportions that structures the inside of a right triangle and at once describes how the right triangle grows and collapses.

Let us explore this process of discovery in diagrams. For it is by these diagrams that Thales discovered and became convinced of all this. Recall the discussion, earlier, of VI.31 and the geometric mean. The mean proportional is an expression of length and areal ratios and equivalences. When the hypotenuse of the right triangle is partitioned equally and unequally by the perpendicular depending upon where on the semi-circle the vertex is located, nevertheless all form right triangles. When the perpendicular divides the hypotenuse un-equally, the longer part stands to the shorter part in a duplicate ratio; of course it does so as well in the isosceles right triangle where the partition is 1:1, but that it stands in a duplicate ratio only became clearer as he divided the right triangles un-equally. Let us follow through the equal and unequal divisions. We can imagine a rectangle made from both lengths as its sides – a rectangle whose sides are the lengths of both extremes. The area of this rectangle is equal to the area of the square drawn on the perpendicular that divides them, and thus the area of the square on the first side stands in duplicate ratio to *both* the square on the perpendicular and the rectangle composed from the first and third side lengths, below:

Geometric Mean

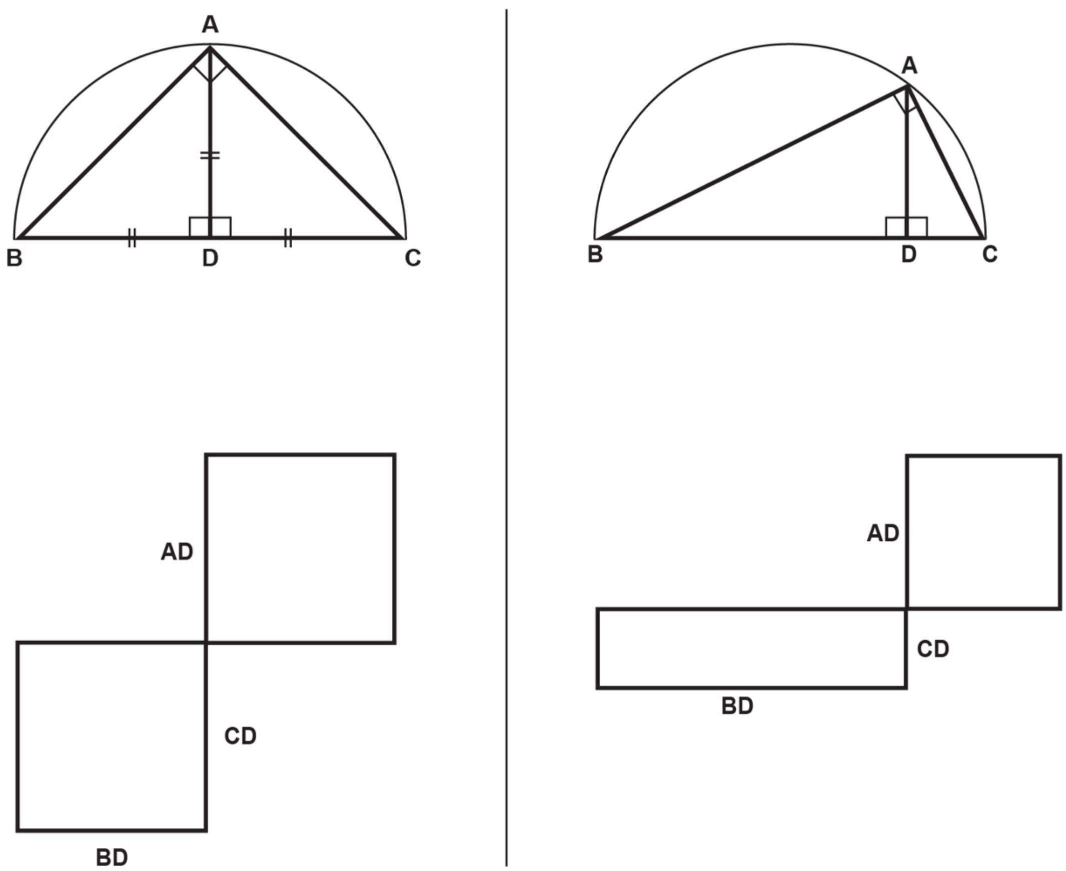


Fig. 39

When the series of connected ideas is presented in Euclid Book VI, the series begins with VI.8 that a perpendicular from the right angle to the hypotenuse always divides the right triangle into two right triangles, each similar to the other and both to the whole largest triangle into which they have been divided, and (Porism) that the perpendicular is the mean proportional. Next, VI.11, to construct a third proportional, three lengths that stand in continuous proportion, then VI.13, the mean proportional of the lengths into which the hypotenuse has been partitioned is constructed by any perpendicular from the diameter of the semi-circle to its circumference; the perpendicular is extended upward from the diameter, and the moment it touches the circumference, it *is* the mean proportional between the two lengths into which the diameter is now partitioned. When we connect that circumference point to the two ends of the diameter, the triangle formed must be right. Next, VI.16, if four lines are proportional the rectangle formed by the longest and shortest lengths (= extremes) is equal in area to the rectangle made of the two middle lengths (= means). Next, VI.17, *if three lines are proportional, the rectangle made by the longest and shortest lengths (= extremes) is equal in area to the square made on the middle length (= means) – here is* *the conclusion to the mean proportional, the three making a continuous proportion*. Next, VI.19, similar triangles are to one another in the duplicate ratio of their corresponding sides, and finally VI.20, All polygons reduce to triangles, and similar ones are to one another in the duplicate ratio of their corresponding sides. Let me set these out again in the context of how we might imagine Thales' discovery. My case is that in the distant past, Thales stumbled upon these ideas in their nascent form as he was looking for the fundamental geometrical figure. Once he realized that it was the right triangle he explored further to see how it grows into all other appearances. The interior of the right triangle dissolves into the same structure into which it expands. Here was his insight into *how* structurally a single underlying unity could come to appear so divergently: this is what I believe Aristotle is referring to when he claims that the Milesians claimed that all appearances alter without changing. I am not arguing that Euclid's proofs mirror Thales, or that his project got farther into constructing the regular solids that are built out of right triangles but rather that it is plausible that Thales had connected a series of mathematical intuitions that supported a new metaphysical project – the construction of the cosmos out of right triangles, and an idea about transformational equivalences to offer an insight into how ὕδωρ could alter without changing.

Let us continue by recalling the diagram that Thales was forced to imagine in measuring a pyramid in Egypt at the time of day when vertical height equals the horizontal shadow (left), and then add to it a matching isosceles right triangle making the larger one into which both divide. Or, since they are similar, simply rotate the same triangle into a different position (right). Again, this is the diagram that appears most in the ancient manuscripts for VI.31.

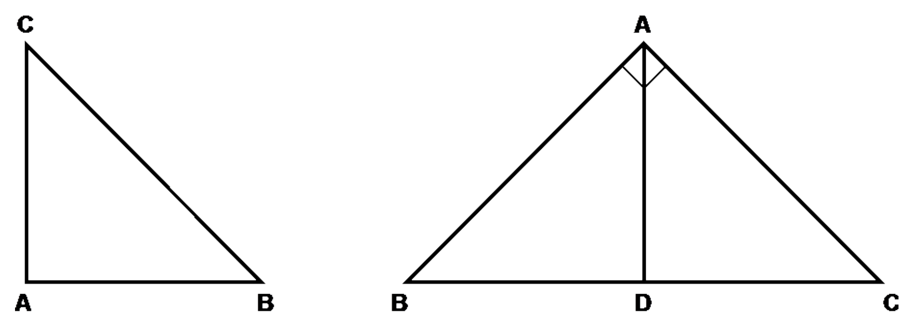


Fig. 40

The *mean proportional* or Geometric Mean is immediately obvious – if you are looking for areal equivalences in a right triangle – in the case of the *isosceles right triangle*. The square on the perpendicular is equal to the rectangle formed by the two segments into which the perpendicular partitions the hypotenuse. It is immediately obvious because each of the segments – AD, BD, CD – are all radii of the circle and hence equal, and so the rectangle made by BD, CD, is an equilateral rectangle or square.

But, the next step for Thales, having discovered that every triangle in the (semi-)circle was right, he wondered whether this areal equivalence held when the perpendicular was begun at some other point along the circle. Do the same areal relations hold there, that is, do they hold for *all* right triangles? If this areal relation held, then the smaller square on AD should still be equal in area to the rectangle BC, CD. He may have suspected this, but what diagrams might have persuaded him and his retinue? By working through the diagrams I am proposing a line of reasoning by which Thales may have thought through the idea of the mean proportional and the idea that when three lines are in continuous proportion, the middle is the mean proportion of the two extremes. Consider the visualization of these relations between square and rectangles, below:

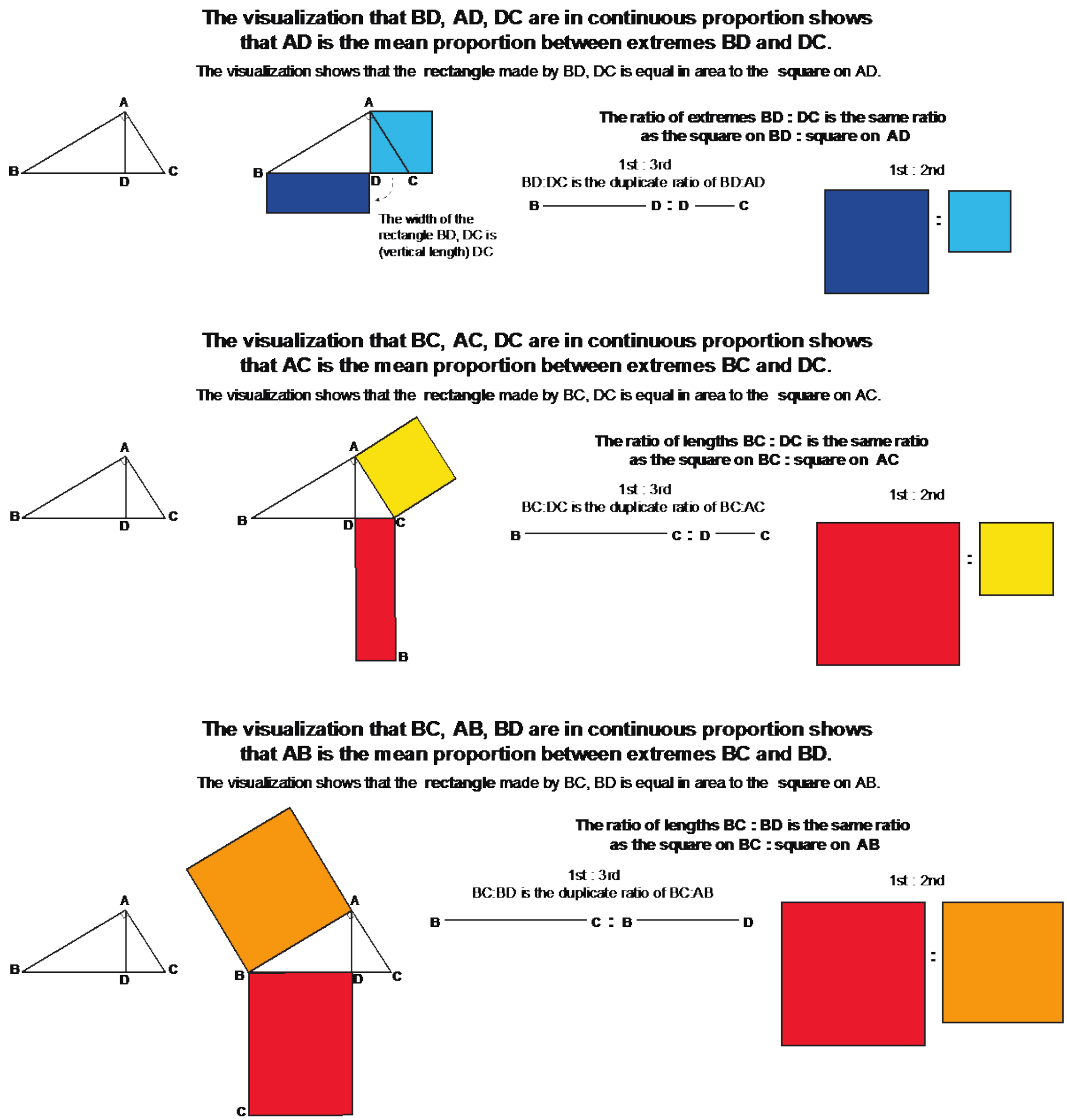


Fig. 41

Thus, once Thales saw the areal equivalence between square on the perpendicular and rectangle made by the two parts into which the hypotenuse was divided, he investigated further to see if there were other areal equivalences. And he found them. Finding them – the two other cases of mean proportionals – was tantamount to discovering an areal interpretation of the Pythagorean theorem. First he thought through the isosceles right triangle that had been so much his focus since the pyramid measurement:

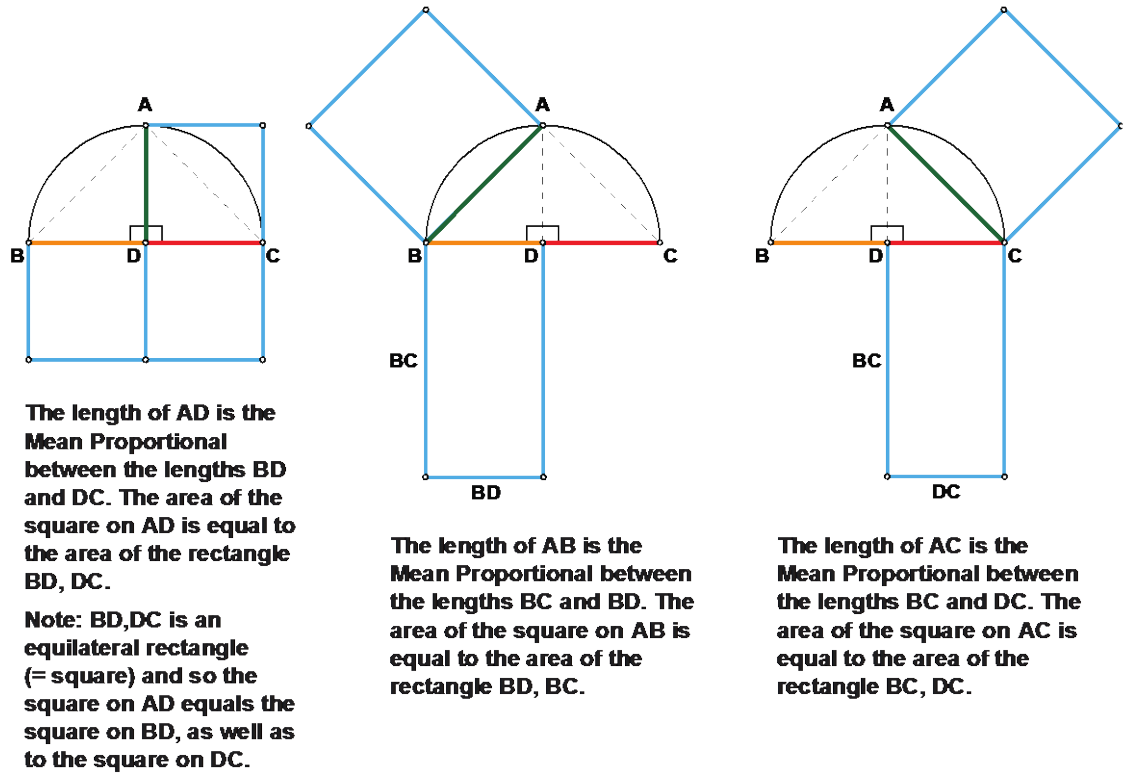
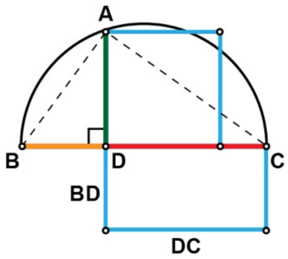
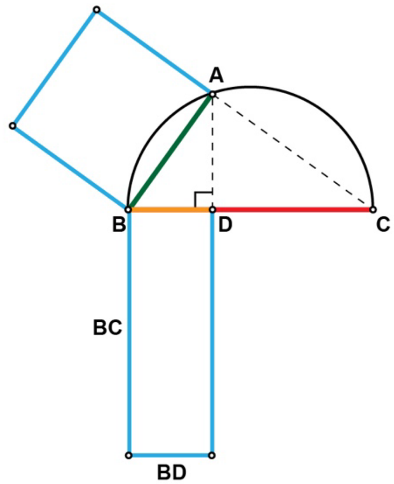


Fig. 42

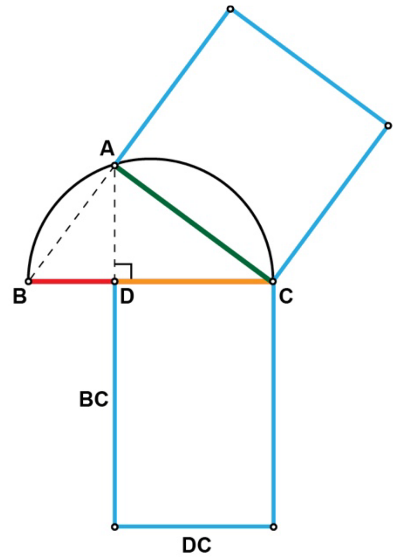
Then, he thought it further with regard to *all* right triangles. While it is certainly true that this intellectual leap is an extraordinary one, it is my argument that these features of a right triangle were explored *because* Thales came to conclude already – or suspected that – the right triangle was the basic building block of all other appearances, that all other appearances could be imagined to be built out of right triangles:



The length of AD is the Mean Proportional between the lengths BD and DC. The area of the square on AD is equal to the area of the rectangle BD, DC.



The length of AB is the Mean Proportional between the lengths BC and BD. The area of the square on AB is equal to the area of the rectangle BC, BD.



The length of AC is the Mean Proportional between the lengths BC and DC. The area of the square on AC is equal to the area of the rectangle BC, DC.

Fig. 43

And thus, he had before him – when the second and third diagrams in each row are combined, is an areal interpretation of the Pythagorean theorem:

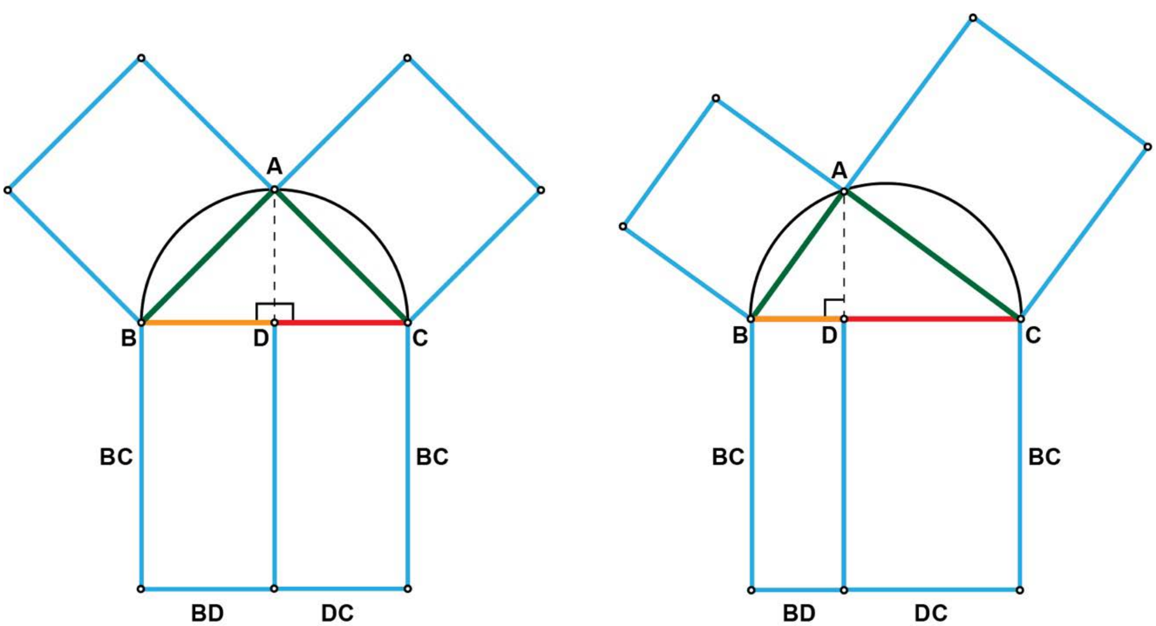


Fig. 44

Next, as I imagine it for Thales and his retinue, to be sure about areal equivalences, he divided the squares by diagonals in the case of the isosceles right triangle (left), and probably at first by measurement, he confirmed the areal equivalences between square and rectangle for scalene triangles (left).

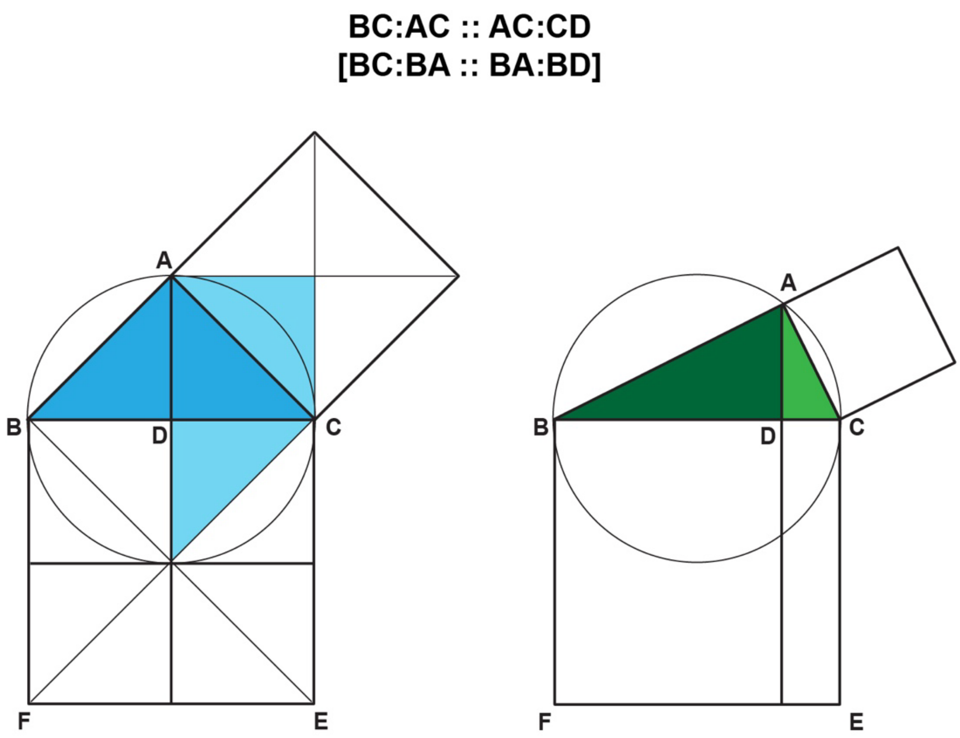


Fig. 45

Stated differently, Thales realized that in addition to perpendicular AD being the mean proportional between lengths BD and CD, he realized that length AB was the mean proportional between diameter BC and segment BD, and thus AC was also the mean proportional between diameter length BC and segment CD. The recognition of these geometric means – "means" of lengths between extremes – amounted to the realization that the area of the square on AB was equal to the rectangle BC,BD, and the area of the square on AC was equal to the rectangle BC, CD. The case of the isosceles right triangle was most immediately obvious (above, left), but dividing the figures into triangles by diagonals allowed him to confirm areal equivalences between squares and rectangles.

Once you know that ABD is *similar* to CAD, you know that BD:AD::AD:DC, for the ratios of corresponding sides must be the same. Stated differently, you know that BD:AD:DC is a continuous proportion. So *by the very definition* of a "mean proportional" AD is a mean proportional between BD and DC. The conclusion of Euclid VI.16, and 17 is that, as a consequence, the rectangle formed by BD,DC is equal to the square on AD. The porism at VI.8 is in effect the analysis for the problem whose synthesis is given in VI.13 – every perpendicular from the diameter of a semi-circle to the circle's circumference is a mean proportional, or geometrical mean, of the lengths into which it divides the diameter. In my estimation, the diagram below, with an isosceles right triangle was the gateway to his realization of the metaphysics of the fundamental geometrical figure that he was looking for.

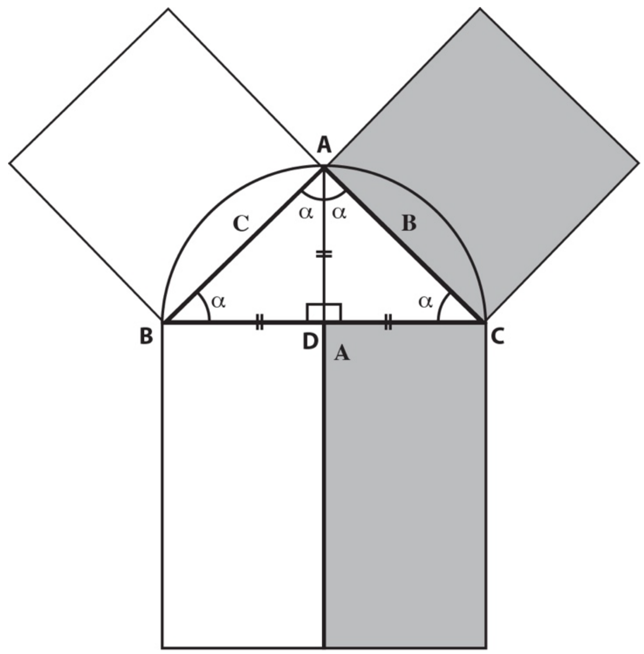


Fig. 46

Thus, the areal relation of the similar and similarly drawn figures constructed on the sides of a right triangle stand in such a relation that the figure on the hypotenuse can be expressed as a square, divisible into two rectangles, and equal in area to the sum of the squares constructed on the two sides, expressed as rectangular parts of the larger square. And this relation holds regardless of the shape of the figures. It applies to *all* shapes, to *all* figures, similar and similarly drawn.

In a review of the scholarly literature I could find no one but Tobias Dantzig who expressed the position that I, too, adopted that Thales knew an areal interpretation of the Pythagorean theorem. Of course, Dantzig acknowledged that his reasons were speculative, and did not find popular support. He put it this way:

> With regard to the Pythagorean theorem my conjecture is that at least in one of its several forms the proposition was known before Pythagoras and that – and this is the point on which I depart from majority opinion – it was known to Thales. I base this conjecture on the fact that the hypotenuse theorem is a direct consequence of the principle of similitude, and that, according to the almost unanimous testimony of Greek historians, Thales was fully conversant with the theory of similar triangles.[74]

And consequently, Dantzig came to believe that "…the similitude proof of [Euclid] Book Six bears the mark of the Founder (of geometry), Thales."[75] Long before I happened upon Dantzig's book, I reached the same conclusion thinking through similitude, but at that point I was still not focused on the fact that there were in Euclid *two* proofs of the hypotenuse theorem, not one. As I continued to gaze upon the diagrams of measuring the height of the pyramid at the time of day when every object casts a shadow equal to its height – and hence focused on isosceles right triangles – having already concluded that Thales was looking for the unity that underlies all three sides of the right triangle – it seemed to me that Thales had already come to some clarity about isosceles triangles, right and otherwise, and that the squares on each of the three sides, each divided by its diagonal, showed immediately that the sum of the areas on the two sides were equal to the area of the square on the hypotenuse. Thales' cosmos is alive, it is *hylozoistic*,[76] it is growing, and collapsing, and it is ruled and connected by geometrical similitude as the structure of its growth. The little world and the big world shared the same underlying structure. And when I read Dantzig's thoughts on Thales, they made an enormous impression because it was from him that I first realized the metaphysical implications of this areal discovery, almost never even considered in the vast secondary literature on the Milesians and early Greek philosophy, and even ancient mathematics. In discussing the Pythagorean theorem and its metaphysical implications, Dantzig claimed:

> To begin with, it [the theorem] is the point of departure of most metric relations in geometry, i.e., of those properties and configurations which are reducible to magnitude and measure. For such figures as are at all amenable to study by classical methods are either polygons or the limits of polygons; and whether the method be *congruence*, *areal equivalence* or *similitude*, it rests, in the last analysis, on the possibility of resolving a figure into triangles.[77]

Thus Dantzig articulated what I, at that stage, had not formulated adequately. The famous theorem is equivalent to the parallel postulate.[78] To grasp one is to grasp the other, though it might not be obvious at first that each implies the other. I am not claiming that Thales' grasped the parallel postulate and all of its implications but rather that he grasped an array of geometrical relations that are exposed when straight lines fall on other straight lines, and how those lines reveal properties of figures, especially in triangles and rectangles. Thus, Thales came to imagine the world as set out in flat space, structured by straight lines and articulated by rectilinear figures. All rectilinear figures can be reduced to triangles, and every triangle can be reduced ultimately to right triangles.

The case I am arguing is that it is either from his inquiries into geometry that Thales came to discover and to grasp an underlying unity – and ἀρχή; or having grasped an underlying unity, came to see geometry as a way to express an unchanging field of relations with a fundamental figure (i.e. the right triangle) that, remarkably, gave insights into the ever-changing world of our experience. This is the legacy of the *practical* diagram. Thales claim that there was an ἀρχή announced that there was an unchanging reality that underlies our experience; his insight into spatial relations through diagrams revealed the structure of it. I have tried to make a plausible case that Thales knew an interpretation of the hypotenuse theorem – the Pythagorean theorem – because, as it turned out, this was what he was looking. Let me be clear that I am arguing that the interpretation of theorem Thales' knew meant two things in the sixth century BCE:

(i) the right triangle was the fundamental geometrical figure
(ii) the right triangle collapses into smaller right triangles, and expands into larger right triangles, by a pattern; we come to refer to that pattern by the expressions "continuous proportions" or "mean proportions" or "geometric means."

> When 3 lines lengths are in continuous proportion, the length of the first is to the length of the third in duplicate ratio, and the figure on the first is to the figure on the second in duplicate ratio.

Euclid preserves at VI.31 a perfected proof of the famous theorem by ratios, proportions, and similar figures. It my contention that this proof is a perfected version of the one grasped by Thales and plausibly proved similarly by Pythagoras and the Pythagoreans. Once Thales was in a position to have learned or confirmed from the Egyptians that all areas of land – all parcels of land divided into rectilinear figures – could be reckoned as the sum of triangles into which they could be dissected, he focused on triangles to see if they revealed some more fundamental figure within them. And what he discovered was that not only did all triangles divide into right triangles but if one continued to divide the right triangles by dropping a perpendicular from the right angle, every right triangle divided into two right triangles, each similar to the other and both similar to the larger triangle. It is not an atomic conception since this process of division into similar triangles was endless. But while there was no smallest right triangle, it was "right triangles – not turtles -- all the way down." And when the right triangles collapse or expand they do so in a pattern that was visualized by the areal equivalence of figures similar and similarly drawn on its perpendicular and its sides.

(1) The perpendicular from the right angle to the hypotenuse divides the hypotenuse into two parts that form a rectangle equal in area to the square on the perpendicular.
(2) The hypotenuse of the largest triangle stands to its shortest side as the hypotenuse of the smallest triangle stands to its shortest side; this proportional relation is visualized as the square on the hypotenuse of the smallest triangle is equal in area to the rectangle made by the hypotenuse and the shortest side.
(3) Again, the hypotenuse of the largest triangle stands to its longest side, as the hypotenuse of the medium size triangle stands to its longest side; this proportional relation is visualized as the square on the hypotenuse of the medium triangle is equal in area to the rectangle made by the hypotenuse and the longest side.

We have three geometrical means exhibited by the right triangle because there are three expressible continuous proportions. What this also entailed was the realization of the relations between line lengths and areas: the ratio of the first and longest length was to the shortest in the same ratio as the figure on the first was to the similar and similarly drawn figure on the second. Thales plausibly knew these relations because this was what he was searching for: the fundamental geometrical figure into which all other figures dissect and out of which all diverse appearances emerge, and moreover to show how a singular underlying unity appears so diversely. The result was that a new program was begun by Thales and embraced and continued by Pythagoras and the Pythagoreans that consisted in the Application of Areas (the transformation of triangles in any angle into parallelograms from one side of the original triangle, how similar figures can be constructed from the original triangle, and the construction of the cosmos out of right triangles by means of the regular solids -- how all appearances could be imagined ultimately to be constructed out of regular equilateral figures – triangles, squares, and pentagons, all of which dissected ultimately into right triangles.

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[1] Some parts of this essay have been adapted from my forthcoming book 2017: *The Metaphysics of the Pythagorean Theorem: Thales, Pythagoras, Engineering, Diagrams, and the Construction of the Cosmos out of Right Triangles.*

[2] Proclus claims that Thales demonstrated (ἀποδεῖξαι, 157.11) that the diameter bisects the circle (= Euclid I.Def. 17) but only notices (ἐπιστῆσαι, 250.22) and states (εἰπεῖν, 250.22) that the angles at the base of an isosceles triangle are equal. Proclus says that Eudemus claims that Thales discovered (εὑρημένον, 299.3) that if two straight lines cut each other, the vertically opposite angles are equal, but he did not prove it scientifically (Euclid I.15). And as we have considered already, Eudemus' claims that Thales must have known that triangles are equal if they share two angles, respectively, and the side between them (= Euclid I.26) because it is a necessary presupposition of the method by which he measured the distance of a ship at sea, that Eudemus accepted as an accomplishment of Thales (Εὔδημος δὲ ἐν ταῖς γεωμετρικαῖς ἱστορίαις εἰς Θαλῆν τοῦτο ἀνάγει τὸ θεώρημα. Τὴν γὰρ τῶν ἐν θαλάττῃ πλοίων ἀπόστασιν δι'οὗ τρόπου φασὶν αὐτὸν δεικνύναι τούτῳ προσχρῆσθαι φησιν ἀναγκαῖον. 352,14-18).

[3] Burkert, Walter. *Lore and Wisdom in Ancient Pythagoreanism*. Trans. E.L. Minar. Harvard University Press, 1972 (first published in the German 1962).

[4] Since Neugebauer's translation of the Cuneiform tablets, we know that the hypotenuse theorem was known in some forms more than a millennium earlier, and likely known earlier elsewhere. My focus is when the Greeks became aware of it in such a way that it found a place within their mathematical reflections, and my emphasis is that for the middle of the 6th century, its place was within the context of metaphysical speculations.

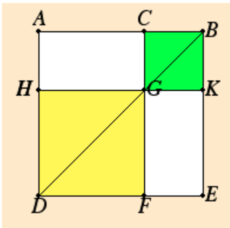
[5] *The Metaphysics of the Pythagorean Theorem*, forthcoming 2017.

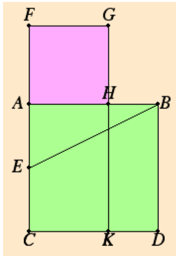
[6] Netz, Reviel. *The Shaping of Deduction in Greek Mathematics*. Cambridge: Cambridge University Press, 1999.
[7] Netz 1999, p. 61.
[8] Netz, Reviel. "Eudemus of Rhodes, Hippocrates of Chios and the earliest form of a Greek mathematical text." *Centaurus* 46: pp. 243-286 2004.
[9] Cf. Kienast, Hermann. *Die Wasserleitung des Eupalinos auf Samos*. Deutsches Archaologisches Institut, Samos, Vol. XIX. Bonn: Dr. Rudoplh Habelt GMBH, 1995.
[10] Cf. Schädler, Ulrich, "Der Kosmos der Artemis von Ephesos", in *Griechische Geometrie im Artemision von Ephesos* ed. Ulrike Muss, 2001, pp. 27.
[11] Netz 1999, p. 60.
[12] Netz 1999, p. 61.
[13] Netz 1999, p. 60.
[14] Von Fritz, Kurt. "The Discovery of Incommensurability by Hippasus of Metapontum," *Annals of Mathematics* 46 (1945), pp. 242-264, esp. 259.
[15] Here I am following the clear outline in Kastanis, Nikos, and Yannis Thomaidis. "The term "Geometrical Algebra," target of a contemporary epistemological debate." users.auth.gr/.../GEOMALGE.pdf
1991.
[16] Neugebauer, Otto. *Exact Sciences in Antiquity*. 2nd Edition. Providence, Rhode Island: Brown University Press, 1957: In his chapter on "Babylonian Mathematics" he writes: "All these problems were probably never sharply separated from methods which we today call 'algebraic.' In the center of this group lies the solution of quadratic equations for two unknowns." (p. 40) Cf. also Rudman, Peter S. *The Babylonian Theorem: The Mathematical Journey to Pythagoras and Euclid.* New York: Prometheus Books. 2010, p. 70: "Such was Neugebauer's stature that his interpretation of that OB scribes were doing quadratic algebra to solve this and other problem texts became the gospel spread by his disciples [esp. van der Waerden and Aaboe]."
[17] In the recent historiography of ancient Greek mathematics, the term "geometrical algebra" was introduced at the end of the 19th century by Tannery, Paul. "De la solution geometrique des problemes du second degree avant Euclide," *Memoires scientifique* 1 Paris, 1912, pp. 254-280, and Zeuthen, H, *Geschichte der Mathematik im Altertum und Mittelalter*, Kopenhagen, Verlag A.F. Hoest, 1896, pp. 32-64, and became current after the publication of Heath's edition of Euclid's *Elements [The Thirteen Books of Euclid's Elements*. Cambridge: Cambridge University Press, 1908. [Dover Reprint, 3 vols., 1956] where he made extensive references to it. The phrase "geometrical algebra" suggested an interpretative approach for reading a number of propositions in Euclid's *Elements*, especially Book II, a book customarily credited to the Pythagoreans. According to those who regarded that the Greeks thought algebraically but decided to systematically organize this thought geometrically, in Book II.4, for example, the algebraic identity $(a+b)^2 = a^2 + 2ab + b^2$ is solved geometrically (below, left, where HD is 'a' and AH is 'b'); and II.11 (below,

right) is considered to be a geometrical solution of the quadratic equation $a(a-x) = x^2$ (where 'x' is AF, and 'a' is AC).

$(a+b)^2 = a^2 + 2ab + b^2$

$x^2 = a(a-x)$





If a straight line be cut at random, then the square on the whole is equal to the sum of the squares on the segments plus twice the rectangle contained by the segments.

To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.

Fig. 47

In the 1930's, Neugebauer's work on ancient algebra supplied a picture of Babylonian arithmetical rules for the solution of quadratic equations, also known as second degree problems,[17] and the earlier hypothesis by Tannery and Zeuthen of geometrical algebra of the Greeks came to be seen as little more than the geometrical formulation and proofs of the Babylonian rules, a point that Neugebauer himself also advocated.[17] This argument, trumpeted by van der Waerden,[17] was that the Greeks were driven to geometrize arithmetic in light of the discovery of incommensurability.

Árpád Szab**ó**, *The Beginnings of Greek Mathematics*. Dordrecht, Holland, and Boston, Mass: D. Reidel Publishing Co, 1978, was the first to break ranks with the hypothesis of geometrical algebra, and hence the view that some parts of Greek geometry were driven to geometrize Babylonian arithmetical rules, by arguing that the propositions of Book II belong to a pre-Euclidean stage of development of Greek geometry that showed a disregard for both proportions and the problem of incommensurability. In the 1970's the debate was propelled further by Sabetai Unguru, "On the Need to Rewrite the History of Greek Mathematics," *Archive for the History of Exact Sciences*, vol. 15, pp. 67-114, 1975, who attacked the historiographic position on geometrical algebra; Unguru argued that it was a mistake to apply modern mathematical techniques to ancient ones in order to discover better how the ancient mathematicians thought through their problems: Geometry is not algebra. As Unguru stated the matter:

> Geometry is thinking about space and its properties. Geometrical thinking is embodied in diagrammatic representation accompanied by a rhetorical component, the proof. Algebraic thinking is characterized by operational symbolism, by the preoccupation with mathematical relations rather than with mathematical objects, by freedom from any ontological commitments, and by supreme abstractedness. 'Geometrical algebra' is not only a logical impossibility it is also a historical impossibility.[17]

Because the centerpiece of the argument for geometric algebra was Euclid's Book II, van der Waerden's reply to Unguru centered on it ("Defense of a 'Shocking' Point of View," *Archive for the History of Exact Sciences*, vol 15, 1975, pp. 199-205). He insisted that (i) *There is no interesting geometrical problem that would justify some of the theorems in Euclid Book II*, (ii) There is a step by step correspondence between the arithmetical methods of the Babylonians and certain theorems of Euclid Book II, and (iii) There are, generally, many points in common to Babylonian and Greek mathematics that are indicative of a transfer of knowledge from the former to the latter.[17]

Fowler, David. "Book II of Euclid's Elements and a Pre-Eudoxian Theory of Ratio," in *Archive for the History of Exact Sciences*, vol 22, pp. 5-36, 1980, countered van der Waerden's assessment of Euclid Book II by arguing that the propositions display techniques for working out ratios using the *anthyphairetic* method, a Pre-Eudoxean method of proportions.[17] Mueller, Ian. *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. New York: Dover Publications. 1981, challenged further the claim that the lines of thought in Book II were algebraic by arguing that since Book II explores the function of geometrical properties of specific figures and not abstract relationships between quantities or as formal relationships between expressions, there is no reason to assume that Euclid is thinking in an algebraic manner.[17] And Berggren, J.L. "History of Greek Mathematics: A Survey of Recent Research," in *Historia Mathematica* 11, 1984, pp. 394-410, argued that in the absence of evidence showing that the Babylonian knowledge was transferred to the Greeks, certain coincidences cannot substantiate such an inference.[17] Knorr, Wilbur, *The Evolution of the Euclidean Elements*. Dordrecht, Netherlands: D. Reidel, 1975. responded to the debate trying to find some areas of compromise. He is clear that the Greeks never had access to algebra, in the modern sense, but they did introduce diagrams not only to explore visually geometric relations but also to apply the results in other contexts. And so, Knorr concluded that "…the term 'geometric algebra' can be useful for alerting us to the fact that, in these instances, diagrams fulfill this function; they are not of intrinsic geometric interest here, but serve only as auxiliaries to other propositions." My position is to acknowledge the use of the term 'geometric algebra' but to caution about its usefulness; it misleads us in understanding how the earliest Greek geometers worked. In my estimation, the Greeks did not have algebra nor was the pattern of their geometrical thought algebraic, though it might well be amenable to algebraic expression. My focus is not on Euclid Book II come the end of the 4th century but rather the earliest forays into geometry in the 6th century that led eventually to it. Szabó and Fowler thought that Book II was a Pre-Eudoxean exploration, and it might well be that by the middle of the 5th century the whole form of thought in what became Book II displayed such a character. But it is my argument that the earliest problem that underlies what later became Book II – as I discuss at length in Hahn 2017, Chapter 3 -- is the squaring of the rectangle, the realization of areal equivalence between a square and a rectangle. This is theorem 14, the last one in the shortest book of Euclid, and I shall detail the lines of thought that constitute the proof later in this book. This is the "interesting geometrical problem" that escaped the notice of van der Waerden; this was the key to grasping continuous proportions, the mean proportional or geometric mean between two line lengths without arguing by ratios and proportions – the pattern by which the fundamental geometrical figure – the right triangle -- scales up and collapses, out of which the metaphysics of the whole cosmos is constructed. The *proof* of the mean proportional was the construction of a rectangle equal in area to a square, and it was that understanding that made it possible for Thales and the early Pythagoreans to grasp the famous hypotenuse theorem, for the internal structure of the right triangle is revealed by the areal equivalences of the figures constructed on its three sides. As we will see in Chapter 3, it is a continued reflection on the triangle in the (semi-)circle, a theorem credited to both Thales and Pythagoras.

[18] Rudman 2010, p. 18. I do not share the same enthusiasm with which Rudman identifies this visualization as "geometric algebra," especially because he thinks the phrase is usefully applied to the Greeks. But what I do share, and which is why I have quoted him, is that visualization of geometric problems through diagrams was a familiar practice from the Old Babylonian period onwards.

[19] After Rudman, p. 69.

[20] After Rudman , p. 19.

[21] As Janet Beery and Frank Sweeney describe this tablet at their website "The Best Known Babylonian Tablet" Home » MAA Press » Periodicals » Convergence » The Best Known Old Babylonian Tablet? "YBC 7289 is an Old Babylonian clay tablet (circa 1800–1600 BCE) from the Yale Babylonian Collection. A hand tablet, it appears to be a practice school exercise undertaken by a novice scribe. But, mathematically speaking, this second millennium BCE document is one of the most fascinating extant clay tablets because it contains not only a constructed illustration of a geometric square with intersecting diagonals, but also, in its text, a numerical estimate of 2‾√ correct to three sexagesimal or six decimal places. The value is read from the uppermost horizontal inscription and demonstrates the greatest known computational accuracy obtained anywhere in the ancient world. It is believed that the tablet's author copied the results from an existing table of values and did not compute them himself. The contents of this tablet were first translated and transcribed by Otto Neugebauer and Abraham Sachs in their 1945 book, *Mathematical Cuneiform Texts* (New Haven, CT: American Oriental Society). More

recently, this tablet was the subject of an article by David Fowler and Eleanor Robson [3], which provides insights into the probable methodology used to obtain such an accurate approximation for 2‾√."

[22] Zhmud, Leonid. *Pythagoras and the Early Pythagoreans*. Oxford: Oxford University Press, 2012.

[23] Zhmud 2012, pp. 242-246.

[24] This story is told by Herodotus II.154 and concerns how the Saite Pharaoh Psammetichus I (Psamtik) (c. 664-610) of the Twenty-Sixth dynasty of Egypt overthrown and in desperation, seeking the advice of the Oracle of Leto at Buto who answered that he should have vengeance when he saw men of bronze coming from the sea." But, when a short time after, it was reported to Psamtik that there were Ionian and Carian pirates who were foraging nearby and wearing bronze armour. Psamtik quickly made friends with them, and seeing in their presence the fulfillment of the oracle, he promised them great rewards if they would join him in his campaign to regain the throne. Upon the success of this endeavor Psmatik kept his promises and bestowed on the mercenaries two parcels of land (or "camps" στρατόπεδα), opposite of each other on either side of the Nile near to his capital in Sais.

[25] Herodotus II.154, 6-8.

[26] Herodotus II.154, 9-11.

[27] Lloyd, Alan B. *Herodotus Book II, Introduction*. Leiden: Brill, 1975, pp. 52-53, voiced just this view of Egyptian influence on what the Greeks could have imported from Egypt.

[28] Isler, Martin. *Sticks, Stones, & Shadows: Building the Egyptian Pyramids*. Norman, Oklahoma: Oklahoma University Press, 2001, pp. 135ff.

[29] Robins, Gay, *Egyptian Statues*. Buckinghamshire, U.K.: Shire Publications Ltd. 2001, p. 60 emphasizes cases where paint splatter can still be detected.

[30] The figure on the right, after Robins, Gay, *Proportion and Style in Ancient Egyptian Art*. London: Thames and Hudson, 1994, p. 161. The photograph on the left taken by the author.

[31] Robins 1994, p. 162.

[32] Bryan, Betsy M. "Painting techniques and artisan organization in the Tomb of Suemniwer," in *Colour and Painting in Ancient Egypt*, W.V. Davies (ed.) London: British Museum Press, 2001, pp. 63-72 2001, p. 64.

[33] Senenmut, perhaps the lover of Queen Hatshepsut, belongs to the 18th dynasty [early new Kingdom], and his tomb chapel [SAE 71] is near to Hatshepsut's. This ostrakon was found in a dump heap just below Senenmut's tomb chapel, 18th dynasty.

[34] After an artifact in the British Museum; here we have a carefully outlined figure of Tuthmosis III. Robins speculates that it might have been a prototype for seated figures of a king in monumental scenes, Robins 2001, p. 60 and plate 15.1.

[35] This useful illustration after www.pyramidofman.com/proportions.htm and also http://library.thinkquest.org/23492/data/. One can see the standing figure is divided into 18 equal units or squares starting from the soles of the feet to the hairline. The bottom of the feet to the knee is between 5 and 6 squares. The elbow line at 2/3 height appears at square 12, and so on. The seated figure occupies 14 squares from soles of the feet to the hairline, etc. See also Robins 1994, p. 166ff.

[36] Cf. Hahn, Robert, *Anaximander in Context: New Studies in the Origins of Greek Philosophy*, Co-Authored with Dirk Couprie and Garard Naddaf), Ancient Philosophy series, Albany, New York: State of New York University Press, 2003. p. 126; Iverson 1975, plate 23, and Robins 1994, p. 161.
[37] Robins 2001, p. 60; cf also Robins 1994, p. 182.
[38] Robins 2001, p. 61.
[39] Robins 1994, p. 63; also pp. 177-181.
[40] Peet 1926, p. 420.
[41] Robins and Shute 1987. Cf. the chapter entitled "Rectangles, Triangles, and Pyramids."
[42] Peet, Thomas Eric. *The Rhind Mathematical Papyrus*. London and Liverpool: University of Liverpool Press, 1923. 1923, p. 91.
[43] The royal cubit consisted of 7 palms, not 6, and is described as the distance between a man's elbow and the tip of his extended middle finger. Its usual equivalence is 52.4 centimeters. One hundred royal cubits was called a khet. The *aroura* was equal to a square of 1 khet by 1 khet, or 10,000 cubits squared. Herodotus tells us that each man received a square parcel of land, but while we know them as *arouras* (= rectangles) the expression of these plots in terms squares 1 khet by 1 khet might explain this discrepancy.
[44] Imhausen, Anette. *Ägyptische Algorithmen. Eine Untersuchung zu den mittelägyptischen mathematischen Aufgabentexten*, Wiesbaden, 2003, pp. 250-251.
[45] Clagett, Marshall. *Ancient Egyptian Science: A Source Book*. 3 volumes. Philadelphia: American Philosophical Society, vol. 3, 1999, p. 163.
[46] Peet, Thomas Eric, *Mathematics in Ancient Egypt*. Manchester, England: Manchester University Press, 1931, p. 432.
[47] Peet 1931, p. 432.
[48] Here I am following the lead of Brunes, Tons. *The Secrets of Ancient Geometry and its use*. Copenhagen, Denmark: Rhodos Publ. 1967, pp. 218-220, but without getting entangled with the broader theory of the origins of geometrical thinking upon which he speculates. I am following him by placing the geometric problems on a square grid, as he does, but he never acknowledges or suggests the grid technique as a familiar process of making large scale paintings for tombs and temples. I am also following some of his written descriptions of how the problems were reckoned.
[49] Robins 1994, p. 57 where, discussing Egyptian statues, she emphasizes the importance of the grid system used by Egyptian sculptors.

[50] Peet 1923, p. 94.
[51] Clagett 1999, vol 3, p. 164.
[52] See my essay "Heraclitus, Milesian Monism, and the Felting of Wool," forthcoming in *Heraklit im Kontext*, Walter de Gryter, Berlin and New York 2017.
[53]Heidel, Joachim P. *Antike Bauzeichnungen*, Darmstadt: Wissenschaftliche Buchgesellschaft, 1993.
[54] Schädler, Ulrich, "Der Kosmos der Artemis von Ephesos", in *Griechische Geometrie im Artemision von Ephesos* ed. Ulrike Muss, pp. 279-287.
[55] I acknowledge gratefully some of the phrasing here concerning these roof tiles by Ulrich Schädler who clarified some of these points in correspondence with me.
[56] The technical German expression is *Kreissegment*.
[57] Schädler 2001, which he identifies as Abb.7: *Projektion der Reconstruktion auf das Blattstabfragment*.
[58] Kienast, Hermann, *The Aqueduct of Eupalinos on Samos*. Athens, Greece: Ministry of Culture, Archaeological Receipts Fund 2005, p. 37.
[59] Herodotus 3.60.
[60] Kienast 1995, p. 172; cf also pp. 150-155. "Es kann kein Zweifel bestehen, daß vor der Inangriffnahme dieser Korrektur eine genau Bauaufnahme der beiden Stollen erarbeitet worden war, und es kann auch kein Zweifel bestehen, daß anhand dieser Bestandspläne der weitere Vortrieb überlegt wurde, auf daß mit möglichst geringem Aufwand mit möglichst hoher Sicherheit der Durchstoß gelänge." I am including here a series of quotations from his report as he argued that Eupalinos must have made a scaled-measured plan:
p. 150
Das Besondere dieses Systems liegt darin, dass die gemessene Einheit klar benannt ist und vor allem natürlich auch darin, dass es mit seiner Zählung von außen nach innen den Vortrieb des Tunnels dokumentiert.

> "The most special feature about this system is that the units of measurement are clearly defined and that it documents the digging of the tunnel from the outside to the inside."

p. 166
Da das Dreieck durch die oberirdisch gemachten Beobachtungen nur in groben Zügen festgelegt war, die genaue Größe aber ausschließlich im Ermessen des Baumeisters lag, kann diese Koinzidenz kein Zufall sein – sie zeigt vielmehr, dass die gewählte Form des Dreiecks mit Maßstab und Zirkel am Zeichentisch erarbeitet wurde.

> "This cannot be a mere coincidence. The triangle was only defined broadly according to the observation above ground. The exact size, however, could only be defined at the discretion of the architect. Thus, it indicates that the chosen form of the triangle was constructed with a scale and a compass at the drawing table."

p. 167

Zurückzuführen lässt sich dieser Fehler am ehesten auf die Eupalinos zur Verfügung stehenden Zeichengeräte und vor allem auf den starken Verkleinerungsmaßstab, mit dem er seinen Tunnelplan gezeichnet haben wird.

> "This mistake is most likely attributable to the drawing devices which were available for Eupalinos and especially to the immense reduction of the scale with which Eupalinos had apparently drawn his tunnel plan." Kienast goes on to argue that when analyzing the original and the new measurement it became obvious that the Λ in one measuring system has shifted by 3 m against the K of the other system. Thus, Eupalinos had assumed the additional digging length resulting from the triangle detour to be 17.60 m. Nevertheless, the additional length that resulted from a triangle with a ratio of 2:5 and a total length of both legs being 270 m was 19.20 m. The difference of 1.60 m between these two numbers amounts to 8 % and is therefore relatively high. However, at this distance, this was scarcely of any importance. Thus, this mistake is most likely attributable to the drawing devices which were available for Eupalinos and especially to the immense reduction of the scale with which Eupalinos had apparently drawn his tunnel plan.

p. 165

Die beiden Stollen werden nicht als gleichwertig pendelnde Größen behandelt, sondern einer wird zum Fangstollen, auf den der andere auftrifft, und die Längen der T-Balken sind eine klare Vorgabe für die einkalkulierte Fehlerquote; sie werden nur – und erst dann – verlängert, wenn der Durchschlag nach Ablauf der Sollzeit immer noch nicht erfolgt ist.

> "The two tunnels were not considered equal but one of them became the "Fangstollen" that the other one was supposed to meet. The lengths of the I-beams were a clear guideline for the calculated error rate; they were only lengthened if the meeting of both tunnels would not occur after the passing of the calculated required time."

p.170

Wie die tatsächlichen Fehler gemessen und verifiziert wurden, wie die Summe der Abweichungen in Länge und Richtung in der Praxis ermittelt wurde, lässt sich nicht mehr feststellen. Erzielen lässt sich dergleichen am ehesten mit einer regelrechten Bauaufnahme, bei der die Messergebnisse vor Ort und maßstäblich auf einem Plan aufgetragen werden.

> "How the actual errors were measured and verified, and how the amount of the deviations of the length and direction were calculated cannot be determined anymore. This was most probably achieved by making an architectural survey for which the measurement results were added on a plan on-site and true to scale."

p. 170:

> "When analyzing the original and the new measurement it became obvious that the Λ in one measuring system has shifted by 3 m against the K of the

other system. Thus, Eupalinos had assumed the additional digging length resulting from the triangle detour to be 17.60 m. Nevertheless, the additional length that resulted from a triangle with a ratio of 2:5 and a total length of both legs being 270 m was 19.20 m. The difference of 1.60 m between these two numbers amounts to 8 % and is therefore relatively high. However, at this distance, this was scarcely of any importance. This mistake is most likely attributable to the drawing devices that were available for Eupalinos and especially to the immense reduction of the scale with which Eupalinos had apparently drawn his tunnel plan."

[61] Hahn 2003, pp. 105-121.

[62] Kienast 2005, p. 49.

[63] Vitruvius III,3.7.

[64] This is a central thesis of all three studies – Hahn, Robert, *Anaximander and the Architects: The Contributions of Egyptian and Greek Architectural Technologies to the Origins of Greek Philosophy*, Ancient Philosophy series, Albany, New York: State of New York University Press, 2001; Hahn, Robert, *Archaeology and the Origins of Philosophy*. Ancient Philosophy series, Albany, New York: State of New York University Press, 2010, and Hahn, Robert 2003.

[65] Hahn 2001, ch 2; 2003, pp. 105-109.

[66] Cf. Kienast 2005, p. 47; on the *eastern* wall there are some 400 different measure marks, painted in red, belonging to other marking systems that Kienast identifies with controlling the depth of the water channel.

[67] The inscription is bordered by two vertical lines that give a distance of 17,10 m. The difference to be measured at the starting point counts 17,59 m and the distance which can be earned at the two letters Λ (original system) and K (shifted system) is 239,80 - 236,80, that is 3 m. If we subtract 3 m from the average distance of the system, i. e. 20,60 the outcome is 17,60 m. That means, that we have two times the same distance of 17,60 m - and the distance at the inscription differs from this ideal measure some 50 cm. But, note also, there are those like Wesenberg who have suggested, instead, that I PARADEGMA I is intended to indicate the ideal of a strengthening wall, since it is written on one.

[68] Kienast 2005, p. 54.

[69] Wesenberg, Burkhardt, *"Das Paradeigma des Eupalinos," Deutschen Archäologischen Instituts, 122, 2007, pp. 33-49.* had a very different conjecture about PARADEGMA. He conjectured that "PARADEGMA" painted on the strengthening wall might have indicated that this "wall" was an example of how a strengthening wall should be made.

[70] Burkert, Walter, 1962/1972.

[71] Zhmud, Leonid, 2012.

[72] D.L. I.24-25.

[73] Heath, Thomas, *A Manuel of Greek Mathematics*. Oxford: Clarendon Press, 1931. [Dover Reprint 1963], p. 82.

[74] Dantzig, Tobias. *Mathematics in Ancient Greece*, Dover Books, New York, 1954, p. 30.
[75] Dantzig 1954, p. 97.
[76] Cf. Gregory, Andrew. *The Presocratics and the Supernatural: Magic, Philosophy and Science in Early Greece*. London: Bloomsbury T&T Clark, 2013, p. 52, just for the most recent mention and discussion of the Milesians as *hylozoists*, though Gregory prefers the term *panpsychists*. But the point is that that scholarship has recognized for a very long time that the Milesians regarded the whole cosmos as alive.
[77] Dantzig 1954, p. 95.
[78] Dantzig 1954, p. 96: "…the Pythagorean relation between the sides of a right triangle was equivalent to the Euclidean *postulate of parallels*."